The princess and the EPR pair

$\Big|$  $\Big\rangle$ $+$ $\Big|$  $\Big\rangle$

$\sqrt{2}$

or

Entanglement spread, communication complexity and information theory.

Aram Harrow
University of Bristol

April 22, 2010

quantum information basics

	Deterministic	Randomized	Quantum
basic unit of information	bit $\{0,1\}$	distribution $p \in \mathbb{R}^2$ $p_0 + p_1 = 1$	qubit $\lvert\psi\rangle = a\lvert 0\rangle + b\lvert 1\rangle \in \mathbb{C}^2$ $\lvert a\rvert^2 + \lvert b\rvert^2 = 1$
n bits	2^n states	2^n dimensions	2^n dimensions
basic unit of computation	NAND, XOR, etc.	stochastic matrices	unitary matrices
measurement	no problem	Bayes' rule	collapses state
correlation	not defined	$p^{AB}(a,b) \neq p^A(a) \cdot p^B(b)$	entanglement $\lvert\psi\rangle \neq \lvert\alpha\rangle \otimes \lvert\beta\rangle$

entanglement

- An old mystery of quantum theory:
  "[not] one, but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."
  ---Schrödinger, 1935

- Spooky action at a distance
  "This makes the reality of [quantities] P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this."
  --- Einstein, Podolsky and Rosen [EPR], 1935

- canonical form:
  EPR pair

$$|\Phi_2\rangle = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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- Quantum computing exploits the exponential scaling to perform calculations that are hard to simulate classically.

general entangled states

Two-party entanglement:

Alice and Bob share $|\psi\rangle = \sum_{ij} c_{ij} \, |i\rangle \otimes |j\rangle$.

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  $|\psi\rangle$ ~ composition.)

- The { nder LOCC:
  i.e., if

But what if classical communication isn't free?

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Idea:

If Alice and Bob are allowed only local unitaries (LU) then the Schmidt coefficients of their state remain exactly the same.

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1. $|\Phi_2\rangle^{\otimes k}$ and $|\Phi_2\rangle^{\otimes l}$ are only approximately orthogonal.
2. Technically we can only approximately decompose $|\psi\rangle$ into

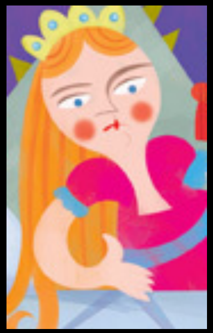
$$\sum_{k\geq 0} \sqrt{p_k}|k\rangle_A|k\rangle_B|\Phi_{\lfloor 2^{\epsilon k}\rfloor}\rangle_{AB}$$

implications

1. Any transformation using local unitaries and Q qubits of communication has off-diagonal blocks decaying as $\leq 2^{Q-\frac{|k-\ell|}{2}}$

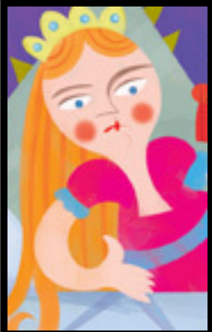
$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

2. 'Exotic' states, such as $|01\rangle^{\otimes n} \pm |\Phi_2\rangle^{\otimes n} / \sqrt{2}$, should be difficult to create, and are potentially valuable.



A bipartite fairy tale

Traditional version: A mysterious woman appears at the castle claiming to be a princess. That night, a single pea placed under twenty mattresses keeps her from sleeping. The prince realises that she is genuine and immediately asks her to marry him.



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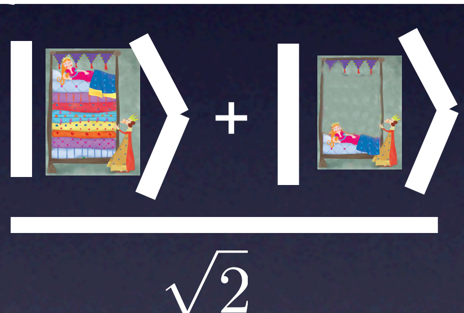

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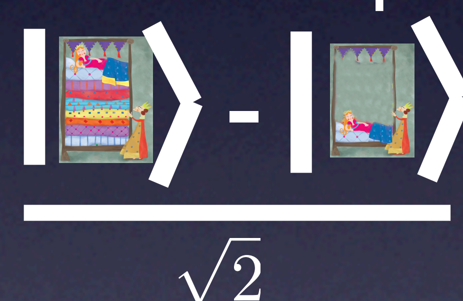

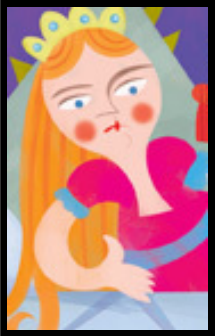

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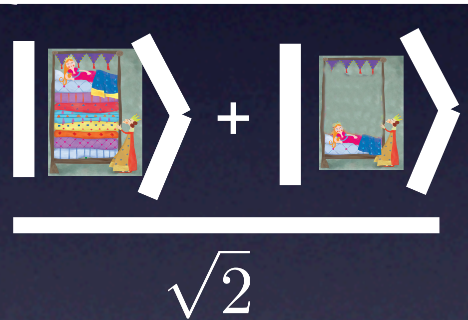

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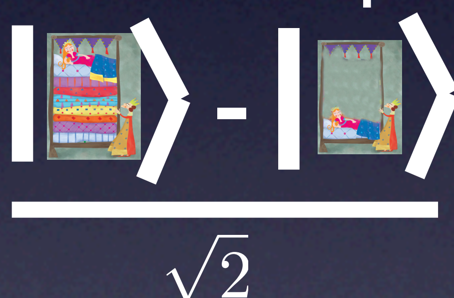
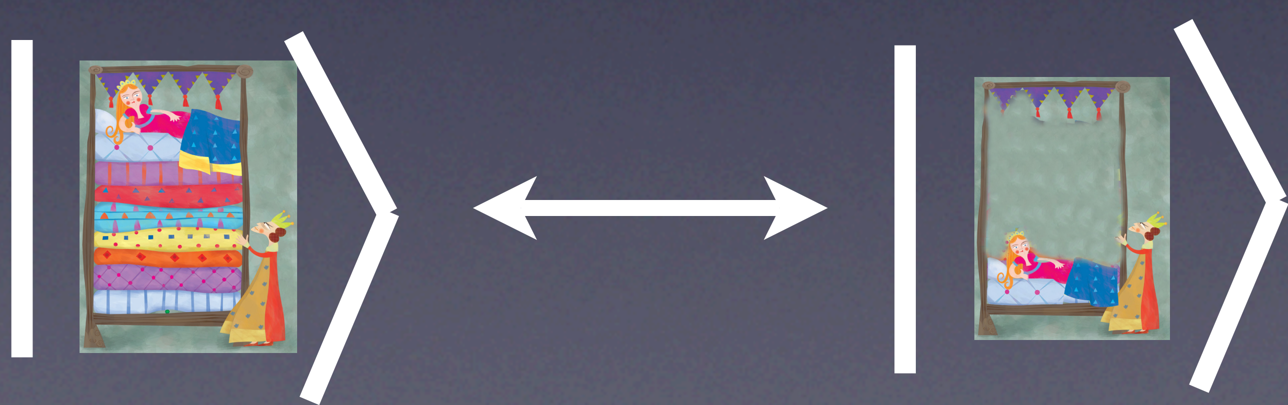


However! Adding or removing lots of mattresses is difficult.



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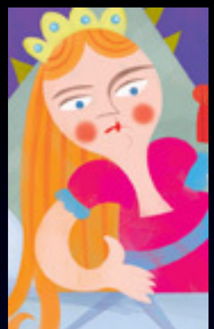
Should he marry her?



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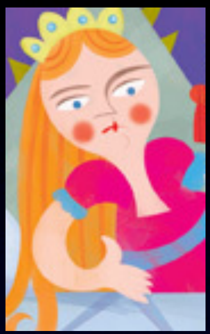

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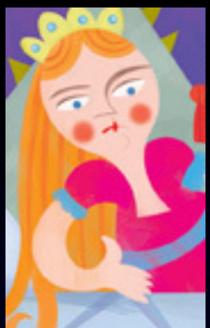

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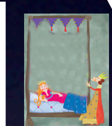
But $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the $\dfrac{|$$\rangle \pm |$$\rangle}{\sqrt{2}}$ basis is equivalent to

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the $|$$\rangle$, $|$$\rangle$ basis.

This performs $|$$\rangle \longleftrightarrow |$$\rangle$

Conclusion: The “princess” is stronger than she looks!

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- This bound holds even given unlimited EPR pairs.

  Why? Because for any m, the same argument applies to the states

  $$|\Phi_2\rangle^{\otimes m} \otimes (|01\rangle^{\otimes n} \pm |\Phi_2\rangle^{\otimes n} / \sqrt{2})$$

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  So the spread of $|\psi\rangle \approx$ the diameter of the support of p.
- Corollary: For $|01\rangle^{\otimes n} + |\Phi_2\rangle^{\otimes n} / \sqrt{2}$, $r\lambda_1 \approx 2^n$. Therefore creating the state requires $\approx n/2$ qubits of communication.

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- However, even in i.i.d. settings, entanglement spread can be size $O(n)$.

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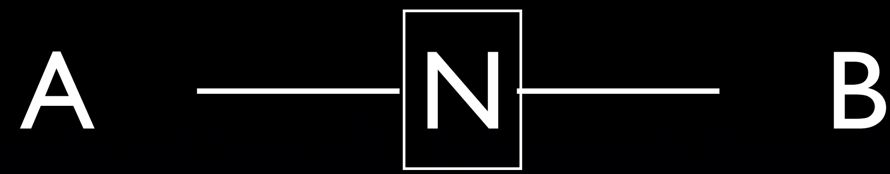

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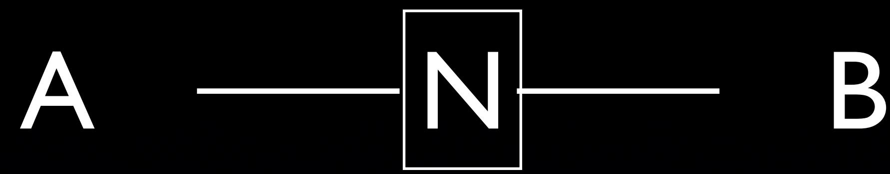

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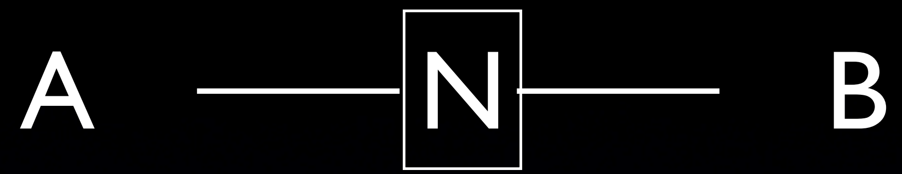

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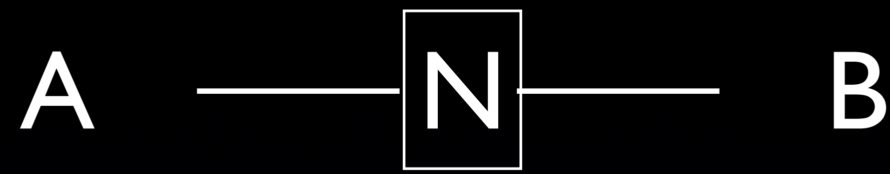

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- However, it does not follow that $\mathcal{N}^{\otimes n}$ can be simulated on arbitrary inputs using $\max_\rho(Q_{\mathcal{N},\rho})$ qubits of communication and $\max_\rho(E_{\mathcal{N},\rho})$ shared EPR pairs!

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- This requires either extra communication (forward or back) or embezzling states.

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- Communication complexity: Special case in which Alice holds $x \in \{0,1\}^n$, Bob holds $y \in \{0,1\}^n$ and they want to compute the bit $f(x,y)$.

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- The proper definition of "free entanglement" is thus closer to "an embezzling state of arbitrary finite size" than "unlimited EPR pairs." In particular, the entangled state in LOSE operations can be taken to be an embezzling state w.l.o.g.

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Problem reduces to projecting onto symmetric subspace.

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  - Simulating U requires O(n) qubits of communication, even using free EPR pairs.
  - With general entanglement, U can be simulated to accuracy ε using $O(\log 1/\varepsilon)$ qubits of communication.
- Corollary: U can asymptotically create O(n) EPR pairs/use, but can only send O(log(n)) bits/use.

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- Free EPR pairs are known to help, although all known examples simply use them to turn classical communication into quantum communication.
- Can non-standard entanglement (e.g. embezzling states) save even more communication?

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Thus we can replace $|\psi\rangle$ with a mixture of states with spread $O(Q/\varepsilon)$ and incur error $\leq \varepsilon$.

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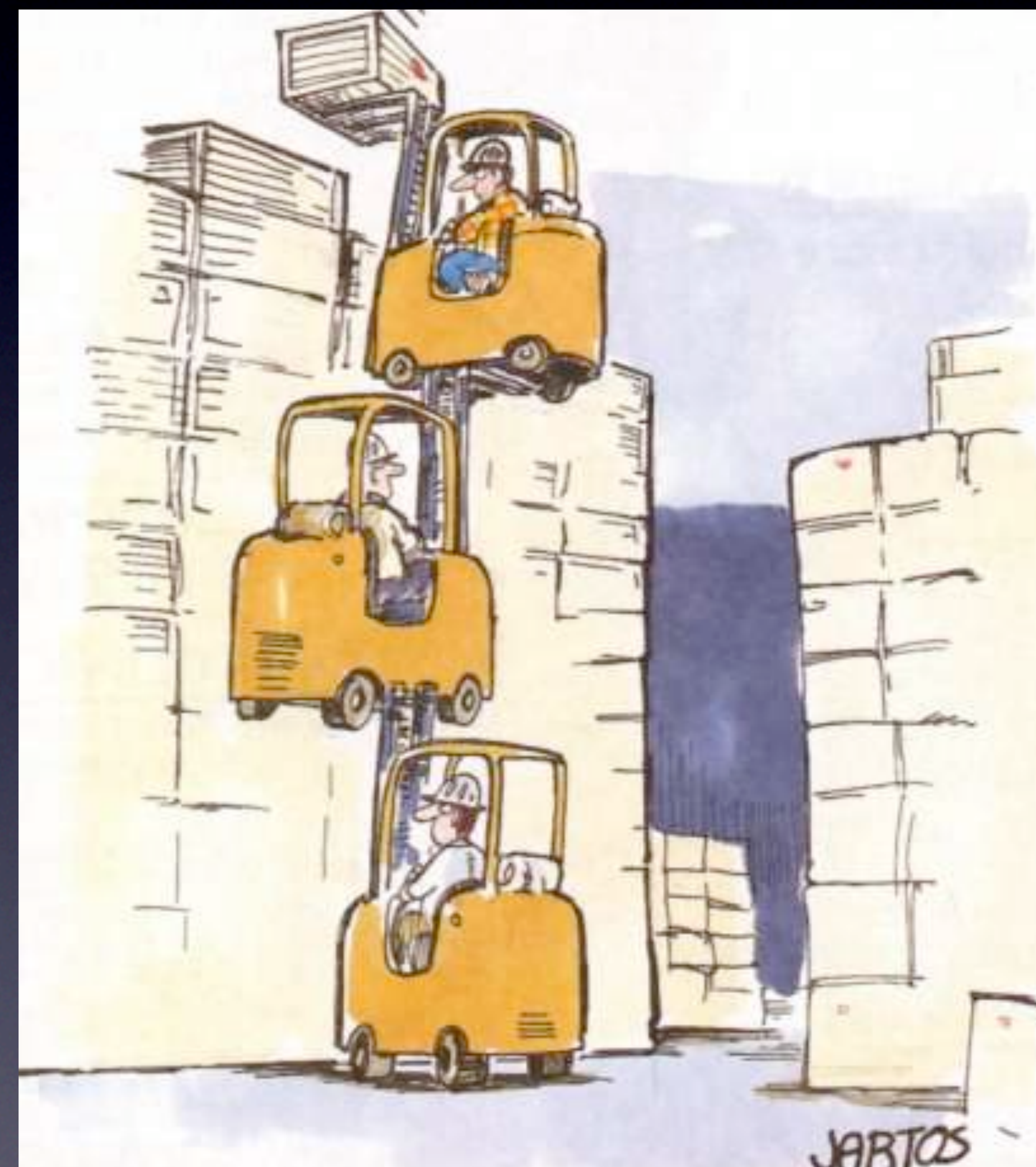
- In communication complexity, how useful even are EPR pairs? Can spread be used to argue that n EPR pairs are not useful for a Q-qubit protocol when $n \gg Q$?

And they all lived happily ever after.



The end.