de Finetti theorems and PCP conjectures

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based on arXiv:1210.6367 + arXiv:13??.????

joint work with Fernando Brandão (UCL)

Symmetric States

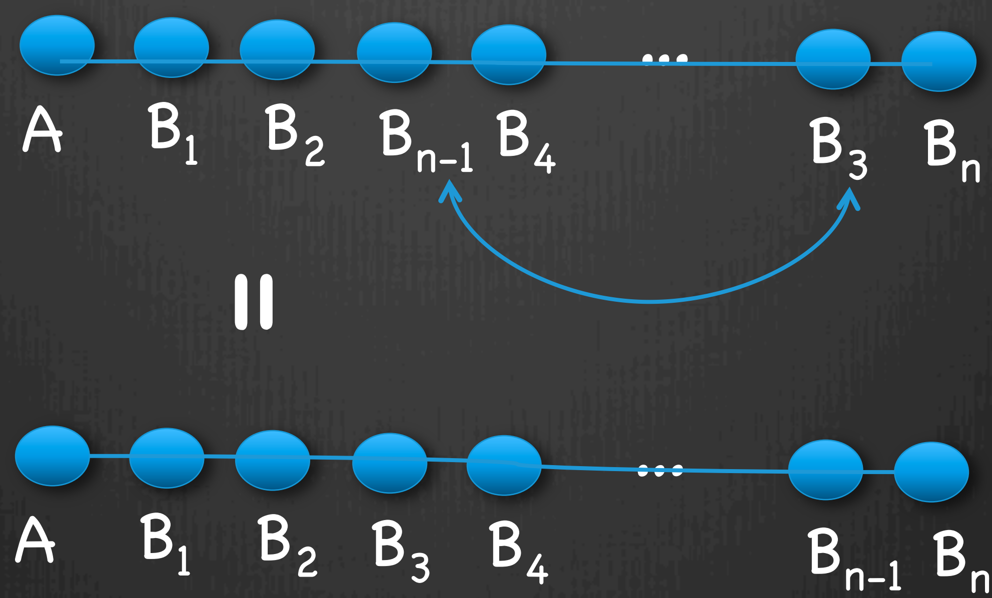
$\rho^{AB_1 \ldots B_n}$ is permutation symmetric in the B subsystems if for every permutation π,

$$\rho^{AB_1 \ldots B_n} = \rho^{AB_{\pi(1)} \ldots B_{\pi(n)}}$$

$\rho^{AB_1 \ldots B_n}$

$=$

$\rho^{AB_1 \ldots B_n}$



Quantum de Finetti Theorem

Theorem [Christandl, Koenig, Mitchison, Renner '06]

Given a state $\rho^{AB_1\ldots B_n}$ symmetric under exchange of $B_1 \ldots B_n$,
there exists μ such that

$$\left\| \rho^{AB_1\ldots B_k} - \int \mu(\mathrm{d}\sigma)\rho_\sigma \otimes \sigma^{\otimes k} \right\|_1 \leq \frac{d^2 k}{n}$$

builds on work by [Størmer '69], [Hudson, Moody '76], [Raggio, Werner '89]
[Caves, Fuchs, Schack '01], [Koenig, Renner '05]

Proof idea:

Perform an informationally complete measurement of n-k B systems.

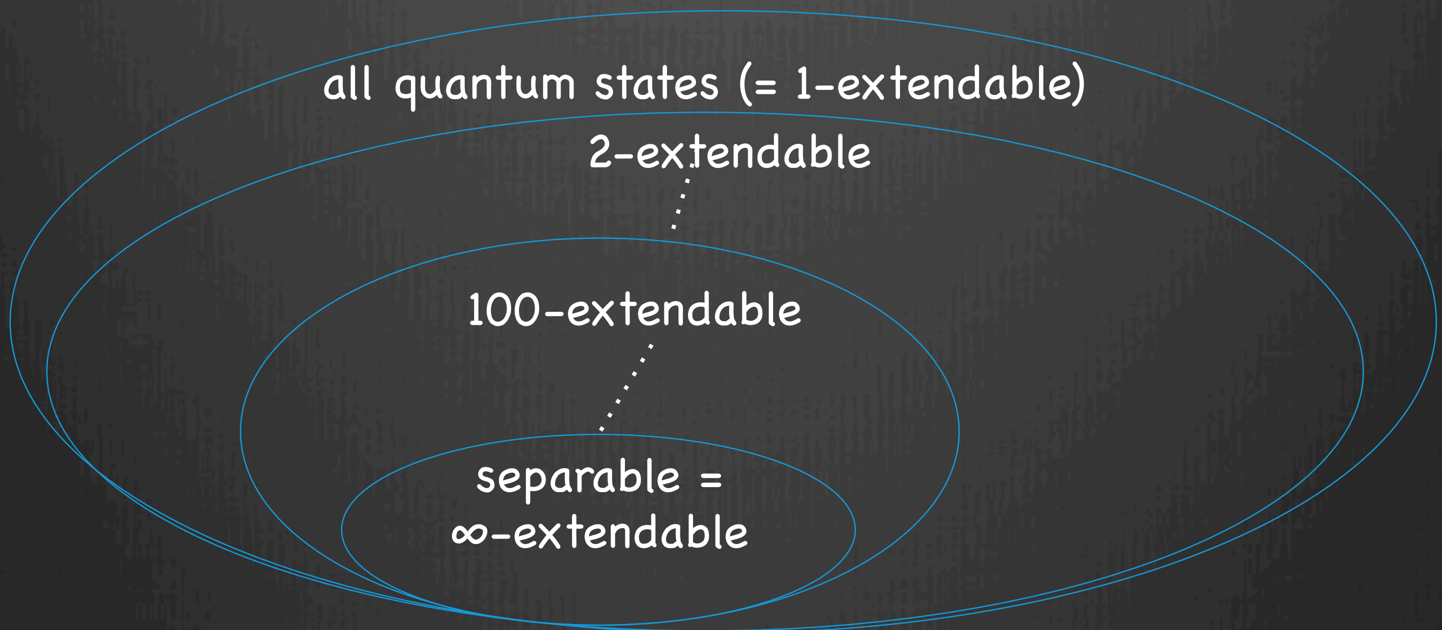
Applications:

information theory: tomography, QKD, hypothesis testing

algorithms: approximating separable states, mean-field theory

Quantum de Finetti Theorem as Monogamy of Entanglement

Definition: ρ^{AB} is n-extendable if there exists an extension $\rho^{AB_1 \ldots B_n}$ with $\rho^{AB} = \rho^{AB_i}$ for each i.



Algorithms: Can search/optimize over n-extendable states in time $d^{O(n)}$.

Question: How close are n-extendable states to separable states?

Quantum de Finetti theorem

Theorem [Christandl, Koenig, Mitchison, Renner '06]

Given a state $\rho^{AB_1\ldots B_n}$ symmetric under exchange of $B_1 \ldots B_n$, there exists μ such that

$$\left\| \rho^{AB_1\ldots B_k} - \int \mu(\mathrm{d}\sigma)\rho_\sigma \otimes \sigma^{\otimes k} \right\|_1 \leq \frac{d^2 k}{n}$$

Difficulty:

1. Parameters are, in many cases, too weak.
2. They are also essentially tight.

Way forward:

1. Change definitions (of error or i.i.d.)
2. Obtain better scaling

relaxed/improved versions

Two examples known:

1. Exponential de Finetti Theorem: [Renner '07]
error term $\exp(-\Omega(n-k))$.
Target state convex combination of "almost i.i.d." states.

2. measure error in 1-LOCC norm [Brandão, Christandl, Yard '10]
For error ε and $k=1$, requires $n \sim \varepsilon^{-2} \log|A|$.

This talk
improved de Finetti theorems for local measurements

main idea

use information theory

log |A| ≥
$I(A:B_1 \ldots B_n) = I(A:B_1) + I(A:B_2|B_1) + \ldots + I(A:B_n|B_1 \ldots B_{n-1})$

repeatedly uses chain rule: $I(A:BC) = I(A:B) + I(A:C|B)$

→ $I(A:B_t|B_1 \ldots B_{t-1}) \leq \log(|A|)/n$ for some $t \leq n$.

If $B_1 \ldots B_n$ were classical, then we would have

$$\rho^{AB} = \rho^{AB_t} = \sum_i \pi_i \rho_i^{AB}$$

≈separable

π_i: distribution on $B_1 \ldots B_{t-1}$

ρ_i^{AB}: ≈product state (cf. Pinsker ineq.)

Question:
How to make $B_{1 \ldots n}$ classical?

Answer: measure!

Fix a measurement $M: B \to Y$.
$I(A:B_t|B_1 \ldots B_{t-1}) \leq \varepsilon$ for the measured state $(\mathrm{id} \otimes M^{\otimes n})(\rho)$.

Then
- ρ^{AB} is hard to distinguish from $\sigma \in \mathrm{Sep}$ if we first apply $(\mathrm{id} \otimes M)$
- $\|(\mathrm{id} \otimes M)(\rho - \sigma)\| \leq$ small for some $\sigma \in \mathrm{Sep}$.

Theorem
Given a state $\rho^{AB_1 \ldots B_n}$ symmetric under exchange of $B_1 \ldots B_n$,
and $\{\Lambda_r\}$ a collection of operations from $A \to X$,

$$\min_{\sigma \in \mathrm{Sep}} \max_{M} \mathbb{E}_{r} \left\| (\Lambda_r^A \otimes M^B)(\rho^{AB} - \sigma^{AB}) \right\|_1 \leq \sqrt{\frac{2 \ln |X|}{n}}$$

Cor: setting Λ=id recovers [Brandão, Christandl, Yard '10] 1-LOCC result.

beware:
X is quantum

the proof

Friendly advice:
You can find these equations in 1210.6367.

$$\pi^{XY_1\dots Y_nR} = \mathop{\mathbb{E}}_r(\Lambda_r^{A\to X}\otimes M_1^{B_1\to Y_1}\otimes\cdots M_n^{B_n\to Y_n})(\rho^{AB_1\dots B_n})\otimes|r\rangle\langle r|^R$$

$$\log|X| \geq \max_{M_1,\dots,M_n} I(X:Y_1\dots Y_n|R)_\pi$$

$$= \max_{M_1,\dots,M_n}\left(I(X:Y_1|R)_\pi+\cdots+I(X:Y_n|Y_1\dots Y_{n-1}R)_\pi\right)$$

$$= \max_{M_1,\dots,M_{n-1}}\Big(I(X:Y_1|R)_\pi+\cdots+I(X:Y_{n-1}|Y_1\dots Y_{n-2}R)_\pi$$

$$+\max_{M_n} I(X:Y_n|Y_1\dots Y_{n-1}R)_\pi\Big)$$

$$= \max_{M_n}\mathop{\mathbb{E}}_r\mathop{\mathbb{E}}_{\vec{y}=(y_1,\dots,y_{n-1})} I(X:Y_n)_{\pi_{r,\vec{y}}}$$

$$\geq \max_M\mathop{\mathbb{E}}_r\mathop{\mathbb{E}}_{\vec{y}}\frac{1}{2}\left\|(\Lambda_r\otimes M)(\rho^{AB}-\rho^A_{\vec{y}}\otimes\rho^B_{\vec{y}})\right\|_1^2$$

$$\geq \min_{\sigma\in\mathrm{Sep}}\max_M\mathop{\mathbb{E}}_r\frac{1}{2}\left\|(\Lambda_r\otimes M)(\rho^{AB}-\sigma^{AB}\right\|_1^2$$

advantages/extensions

<u>Theorem</u>

Given a state $\rho^{AB_1 \ldots B_n}$ symmetric under exchange of $B_1 \ldots B_n$, and $\{\Lambda_r\}$ a collection of operations from A→X,

$$\min_{\sigma \in \mathrm{Sep}} \max_{M} \mathbb{E}_{r} \left\| (\Lambda_r^A \otimes M^B)(\rho^{AB} - \sigma^{AB}) \right\|_1 \leq \sqrt{\frac{2 \ln |X|}{n}}$$

1. Simpler proof and better constants
2. Bound depends on |X| instead of |A| (A can be ∞-dim)
3. Applies to general non-signalling distributions
4. There is a multipartite version (multiply error by k)
5. Efficient "rounding" (i.e. σ is explicit)
6. Symmetry isn't required

applications

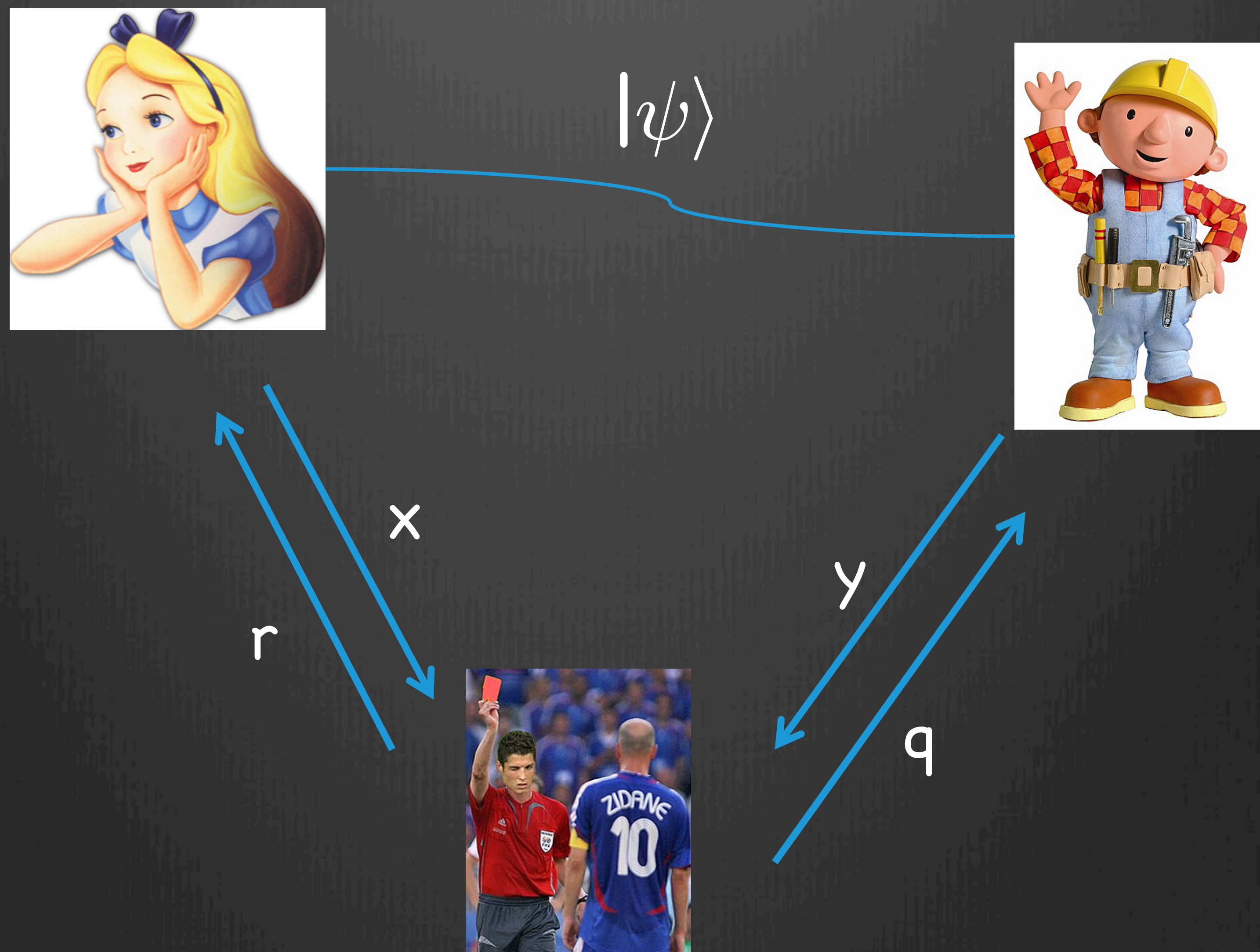
- nonlocal games
  Adding symmetric provers "immunizes" against entanglement / non-signalling boxes. (Caveat: needs uncorrelated questions.)
  Conjectured improvement would yield NP-hardness for 4 players.

- BellQMA(poly) = QMA
  Proves Chen-Drucker SAT $\in$ $\text{BellQMA}_{\log(n)}(\sqrt{n})$ protocol is optimal.

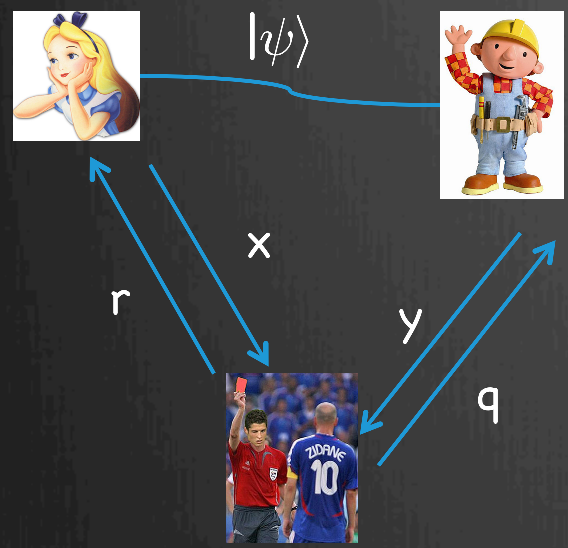
- pretty good tomography [Aaronson '06]
  on permutation-symmetric states (instead of product states)

- convergence of Lasserre hierarchy for polynomial optimization
  see also 1205.4484 for connections to small-set expansion

non-local games



non-local games



Non-Local Game $G(\pi, V)$:

$\pi(r, q)$: distribution on R x Q

$V(x, y|r, q)$: predicate on X x Y x R x Q

Classical value: $$\omega_c(G) = \max_{x,y} \mathop{\mathbb{E}}_{(r,q)\sim\pi} V(x(r), y(q)|r, q)$$

Quantum value: $$\omega_e(G) = \sup \mathop{\mathbb{E}}_{(r,q)\sim\pi} \sum_{x,y} V(x(r), y(q)|r, q) \langle\psi| L_x^r \otimes M_y^q |\psi\rangle$$

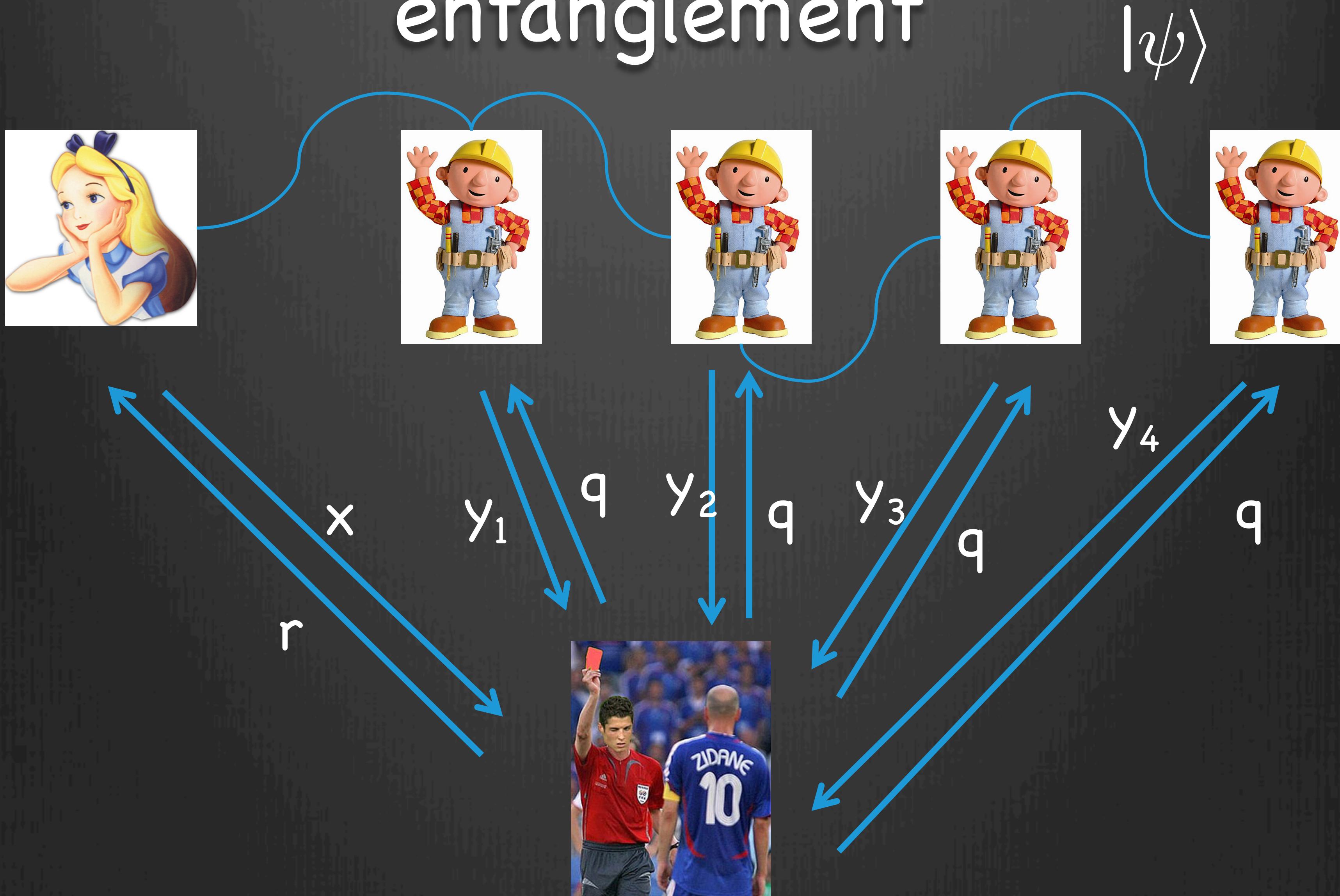
$$\sum_x L_x^r = I \qquad \sum_y M_y^q = I$$

sup over measurements and $|\psi\rangle$ of unbounded dim

previous results

- [Bell '64]
  There exist G with $\omega_e(G) > \omega_c(G)$
- PCP theorem [Arora et al '98 and Raz '98]
  For any $\varepsilon > 0$, it is NP-complete to determine whether $\omega_c < \varepsilon$ or $\omega_c > 1-\varepsilon$ (even for XOR games).
- [Cleve, Høyer, Toner, Watrous '04]
  Poly-time algorithm to compute ω_e for two-player XOR games.
- [Kempe, Kobayashi, Matsumoto, Toner, Vidick '07]
  NP-hard to distinguish $\omega_e(G) = 1$ from $\omega_e(G) < 1-1/\text{poly}(|G|)$
- [Ito-Vidick '12 and Vidick '13]
  NP-hard to distinguish $\omega_e(G) > 1-\varepsilon$ from $\omega_e(G) < \frac{1}{2} + \varepsilon$ for three-player XOR games

immunizing against entanglement



complexity of non-local games

Cor: Let $G(\pi,V)$ be a 2-player free game with questions in $R \times Q$ and answers in $X \times Y$, where $\pi=\pi^R \otimes \pi^Q$. Then there exists an $(n+1)$-player game $G'(\pi',V')$ with questions in $R \times (Q_1 \times \ldots \times Q_n)$ and answers in $X \times (Y_1 \times \ldots \times Y_n)$, such that

$$\omega_c(G) \leq \omega_e(G') \leq \omega_c(G) + \sqrt{\frac{\ln |X|}{2n}}$$

Implies:

1. an $\exp(\log(|X|)\log(|Y|))$ algo for approximating ω_c
2. ω_e is hard to approximate for free games.

why free games?

Theorem

Given a state $\rho^{AB_1\ldots B_n}$ symmetric under exchange of $B_1 \ldots B_n$, and $\{\Lambda_r\}$ a collection of operations from A→X,

$$\min_{\sigma\in\text{Sep}} \max_{M} \mathbb{E}_{r} \left\|(\Lambda_r^A \otimes M^B)(\rho^{AB} - \sigma^{AB})\right\|_1 \leq \sqrt{\frac{2\ln|X|}{n}}$$

$\exists\, \sigma$ $\forall\, q$ for most r ρ and σ give similar answers

Conjecture

Given a state $\rho^{AB_1\ldots B_n}$ symmetric under exchange of $B_1 \ldots B_n$, and $\{\Lambda_r\}$ a collection of operations from A→X,

$$\min_{\sigma\in\text{Sep}} \mathbb{E}_{r} \max_{M} \left\|(\Lambda_r^A \otimes M^B)(\rho^{AB} - \sigma^{AB})\right\|_1 \leq \sqrt{\frac{2\ln|X|}{n}}$$

- Would give alternate proof of Vidick result.
- FALSE for non-signalling distributions.

QCC…C de Finetti

Theorem

If $\rho^{A_1,\ldots,A_n}$ is permutation symmetric then for every k there exists μ s.t.

$$\max_{M_2,\ldots,M_k} \left\| (\mathrm{id} \otimes M_2 \otimes \cdots \otimes M_k)(\rho^{A_1\ldots A_k} - \int \mu(\sigma)\sigma^{\otimes k} \right\|_1 \leq \sqrt{\frac{2k^2 \ln |A|}{n-k}}$$

Applications

- QMA = QMA with multiple provers and Bell measurements
- convergence of sum-of-squares hierarchy for polynomial optimization
- Aaronson's pretty-good tomography with symmetric states

de Finetti without symmetry

Theorem [Christandl, Koenig, Mitchison, Renner '05]

Given a state $\rho^{AB_1\ldots B_n}$, there exists μ such that

$$\left\| \mathop{\mathbb{E}}_{i_1,\ldots,i_k} \rho^{AB_{i_1}\ldots B_{i_k}} - \int \mu(\mathrm{d}\sigma)\rho_\sigma \otimes \sigma^{\otimes k} \right\|_1 \leq \frac{d^2 k}{n}$$

Theorem

For ρ a state on $A_1A_2\ldots A_n$ and any $t \leq n-k$, there exists $m \leq t$ such that

$$\mathop{\mathbb{E}}_{i_1,\ldots,i_k} \mathop{\mathbb{E}}_{\substack{j_1,\ldots,j_m \\ a_1,\ldots,a_m}} \left\| \sigma^{A_{i_1}\cdots A_{i_k}} - \sigma^{A_{i_1}} \otimes \cdots \otimes \sigma^{A_{i_k}} \right\|_1 \lesssim \frac{d^k}{n-k}$$

where σ is the state resulting from measuring $j_1,\ldots,j_m$ and obtaining outcomes $a_1,\ldots,a_m$.

PCP theorem

Classical k-CSPs:
Given constraints $C=\{C_i\}$, choose an assignment σ mapping n variables to an alphabet Σ to minimize the fraction of unsatisfied constraints.

$$\mathrm{UNSAT}(C) = \min_{\sigma} \Pr_i [\sigma \text{ fails to satisfy } C_i]$$

Example: 3-SAT:
NP-hard to determine if UNSAT(C)=0 or UNSAT(C) $\geq 1/n^3$

PCP (probabilistically checkable proof) theorem:
NP-hard to determine if UNSAT(C)=0 or UNSAT(C) $\geq$ 0.1

Local Hamiltonian problem

LOCAL-HAM: k-local Hamiltonian ground-state energy estimation

Let $H = \mathbb{E}_i H_i$, with each H_i acting on k qubits, and $\|H_i\| \leq 1$

i.e. $H_i = H_{i,1} \otimes H_{i,2} \otimes \ldots \otimes H_{i,n}$, with $\#\{j : H_{i,j} \neq I\} \leq k$

Goal:

Estimate $E_0 = \min_\psi \langle\psi|H|\psi\rangle = \min_\rho \operatorname{tr} H\rho$

Hardness

- Includes k-CSPs, so ±0.1 error is NP-hard by PCP theorem.
- QMA-complete with 1/poly(n) error [Kitaev '99]

  QMA = quantum proof, bounded-error polytime quantum verifier

Quantum PCP conjecture

LOCAL-HAM is QMA-hard for some constant error $\varepsilon > 0$.

Can assume k=2 WLOG [Bravyi, DiVincenzo, Terhal, Loss '08]

high-degree in NP

Theorem

It is NP-complete to estimate E_0 for n qudits on a D-regular graph to additive error $\sim d / D^{1/8}$.

Idea: use product states

$E_0 \approx \min \operatorname{tr} H(\psi_1 \otimes \dots \otimes \psi_n) - O(d/D^{1/8})$

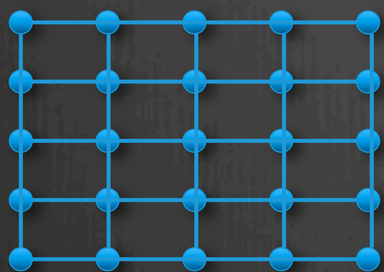
By constrast

2-CSPs are NP-hard to approximate to error $|\Sigma|^\alpha/D^\beta$ for any $\alpha,\beta>0$

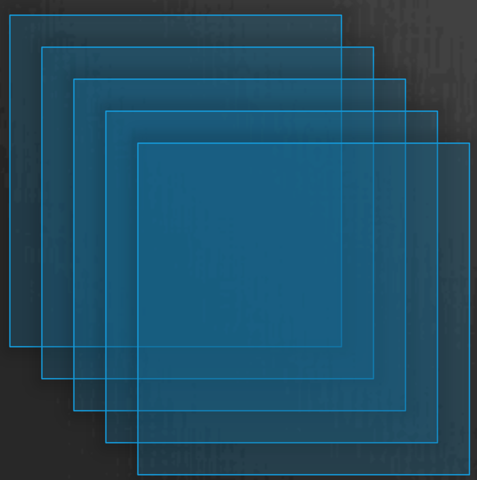
intuition: mean-field theory

1-D

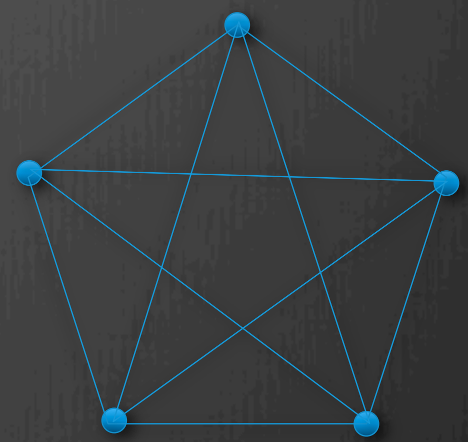
2-D



3-D



∞-D



Proof of PCP no-go theorem

1. Measure εn qudits and condition on outcomes.
   Incur error ε .

2. Most pairs of other qudits would have mutual information
   $\leq \log(d) / \varepsilon D$ if measured.

3. Thus their state is within distance $d^3(\log(d) / \varepsilon D)^{1/2}$ of product.

4. Witness is a global product state. Total error is
   $\varepsilon + d^3(\log(d) / \varepsilon D)^{1/2}$.
   Choose ε to balance these terms.

other applications

PTAS for Dense k-local Hamiltonians
improves on $1/d^{k-1} + \varepsilon$ approximation from [Gharibian-Kempe '11]

PTAS for planar graphs
Builds on [Bansal, Bravyi, Terhal '07] PTAS for bounded-degree planar graphs

Algorithms for graphs with low threshold rank
Extends result of [Barak, Raghavendra, Steurer '11].
run-time for ε-approximation is
$\exp(\log(n)\ \mathrm{poly}(d/\varepsilon) \cdot \#\{\text{eigs of adj. matrix} \geq \mathrm{poly}(\varepsilon/d)\})$

open questions

- Is QMA(2) = QMA? Is SAT$\in$QMA$_{\sqrt{n}}(2)_{1,1/2}$ optimal? (Would follow from replacing 1-LOCC with SEP-YES.)

- Can we reorder our quantifiers to get a dimension-independent bound for correlated local measurements?

- (Especially if your name is Graeme Mitchison) Representation theory results -> de Finetti theorems What about the other direction?

- The usual de Finetti questions:
  - better counter-examples
  - how much does it help to add PPT constraints?

- The unique games conjecture is ≈equivalent to determining whether max {tr Mρ : $\rho \in$ Sep} is $\geq c_1/d$ or $\leq c_2/d$ for $c_1 \gg c_2 \gg 1$ and M a LO measurement. Can we get an algorithm for this using de Finetti?

- Weak additivity? The Quantum PCP conjecture?

arXiv:1210.6367