A counterexample for the graph area law conjecture

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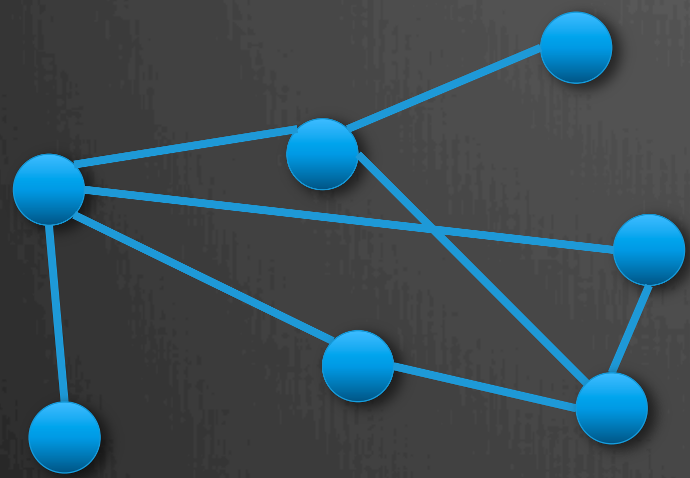
Background: local Hamiltonians



$$H = \sum_{(i,j)\in E} H_{i,j}$$

$\| H_{i,j} \| \leq 1$

Define: eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$
eigenstates $|\psi_0\rangle, |\psi_1\rangle, \ldots$

Assume: degree ≤ const, gap := $\lambda_1 - \lambda_0 \geq$ const.

Known: $|\langle AB\rangle - \langle A\rangle\langle B\rangle| \lessapprox \|A\| \|B\| \exp(-\text{dist}(A,B) / \xi)$
"correlation decay" [Hastings '04, Hastings-Kumo '05, ...]

Intuition: $((1+\lambda_0)I - H)^{O(1)} \approx \psi_0$ $\langle X\rangle := \text{tr}[X\psi_0]$
[Arad-Kitaev-Landau-Vazirani, 1301.1162]

Area "law"?

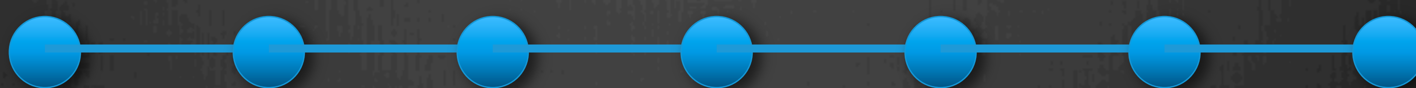
Conjecture: For any set of systems A⊆V

$$S(\psi_0^A) \leq O(|\partial A|)$$

Or even, with variable dimensions d_1, …, d_n.

$$S(\psi_0^A) \leq O(1) \cdot \sum_{\substack{(i,j)\in E \\ i\in A, j\in A^c}} \log(d_i) + \log(d_j)$$

Known: in 1-D

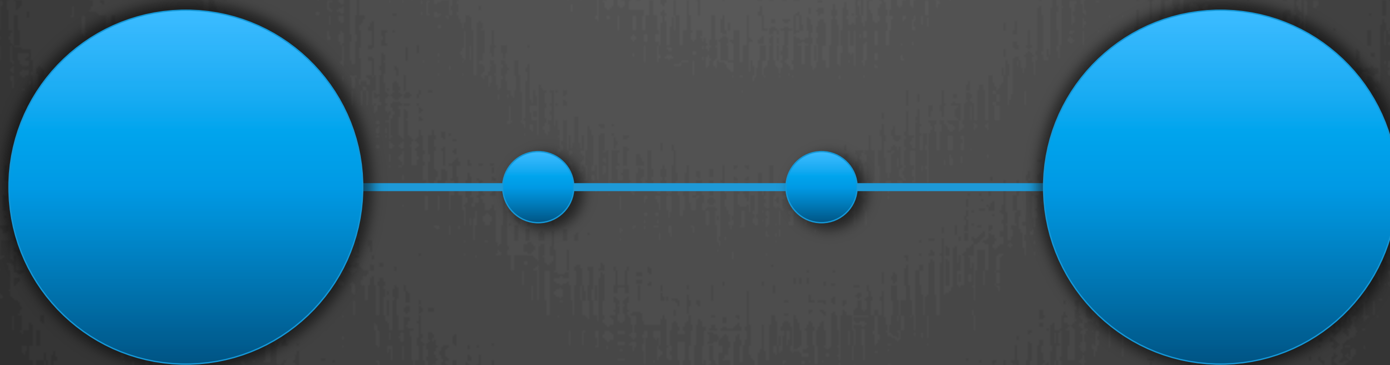


gap —(Hastings '04)→ correlation decay —(Brandão-Horodecki '12)→ area law

gap —(Hastings '07, Arad-Kitaev-Landau-Vazirani '13)→ area law

further implications: efficient description (MPS), algorithms

This talk

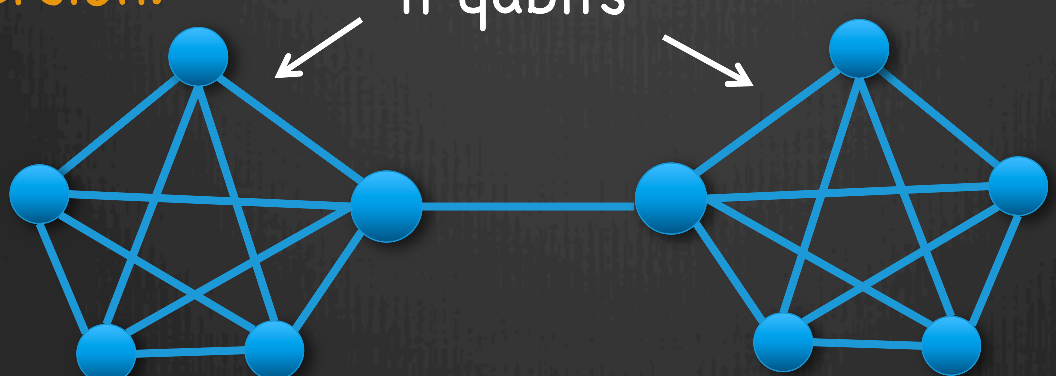


dimension N 3 3 N

Entanglement $\propto \log(N)$

Qubit version:

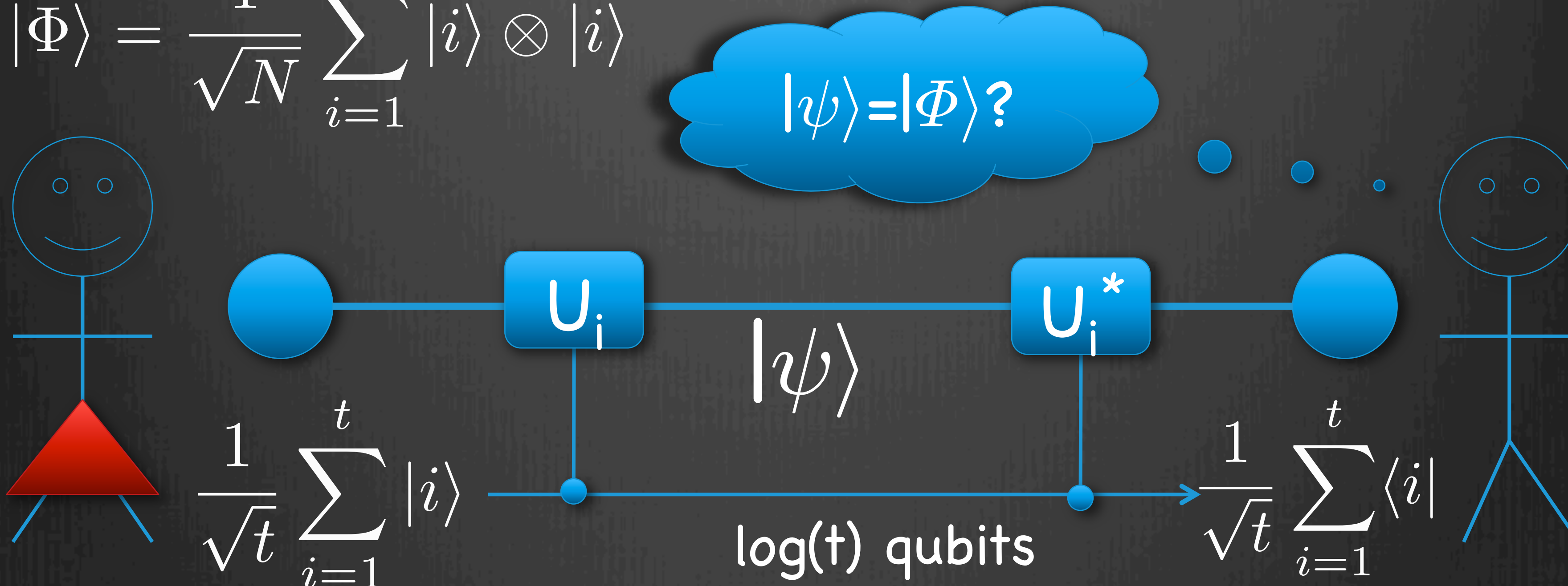
n qubits



entanglement $\propto n^c$

Apparent detour: EPR testing

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$$



Idea: $|\Phi\rangle$ is unique state invariant under $U \otimes U^*$.

Result: Error λ with O(log $1/\lambda$) qubits sent.

Previous work used O(loglog(N) + log(1/ λ)) qubits [BDSW '96, BCGST '02]

EPR testing protocol

steps	state
1. Initial state:	$\vert\psi\rangle^{AB}$
2. Alice adds ancilla in $\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\vert i\rangle^{A'}$ state	$\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\vert i\rangle^{A'}\vert\psi\rangle^{AB}$
3. Alice applies controlled U_i i.e. $\Sigma_i \vert i\rangle\langle i\vert \otimes U_i$	$\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\vert i\rangle^{A'}(U_i\otimes I)\vert\psi\rangle^{AB}$
4. Alice sends A′ to Bob and Bob applies controlled U_i^*	$\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\vert i\rangle^{B'}(U_i\otimes U_i^*)\vert\psi\rangle^{AB}$
5. Bob projects B′ onto $t^{-1/2}\Sigma_i \vert i\rangle$.	$\frac{1}{t}\sum_{i=1}^{t}(U_i\otimes U_i^*)\vert\psi\rangle^{AB}$

Analyzing protocol

Subnormalized output state (given acceptance) is

$$M|\psi\rangle = \frac{1}{t}\sum_{i=1}^{t}(U_i \otimes U_i^*)|\psi\rangle$$

Pr[accept] = $\langle \psi | M^\dagger M | \psi \rangle$

Key claim: $\| M - |\Phi\rangle\langle\Phi| \| \leq \lambda$

Interpretation as super-operators:

$X = \Sigma_{a,b} X_{a,b} |a\rangle\langle b|$ → $|X\rangle = \Sigma_{a,b} X_{a,b} |a\rangle \otimes |b\rangle$

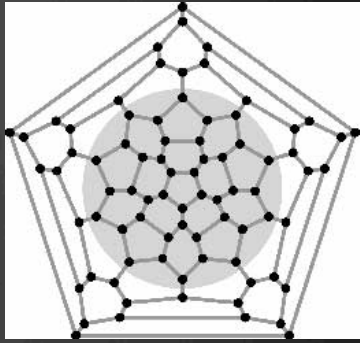
$T(X) = AXB$ → $T|X\rangle = (A \otimes B^T)|X\rangle$

$T(X) = UXU^\dagger$ → $T|X\rangle = (U \otimes U^*)|X\rangle$

identity matrix → $|\Phi\rangle$

$\|M(X)\|_2 \leq \lambda \|X\|_2$ if $tr[X]=0$ ←→ $\| M - |\Phi\rangle\langle\Phi| \| \leq \lambda$

Quantum expanders



A collection of unitaries $U_1, \ldots, U_t$ is a quantum (N, t, λ) expander if

$$\left\| \frac{1}{t} \sum_{i=1}^{t} U_i X U_i^\dagger \right\|_2 \leq \lambda \|X\|_2 \quad \text{whenever } \mathrm{tr}[X]=0$$

(cf. classical t-regular expander graphs can be viewed as permutations $\pi_1, \ldots, \pi_t$ such that $t^{-1} \|\Sigma_i \pi_i x\|_2 \leq \lambda \|x\|_2$ whenever $\Sigma_i x_i = 0$.)

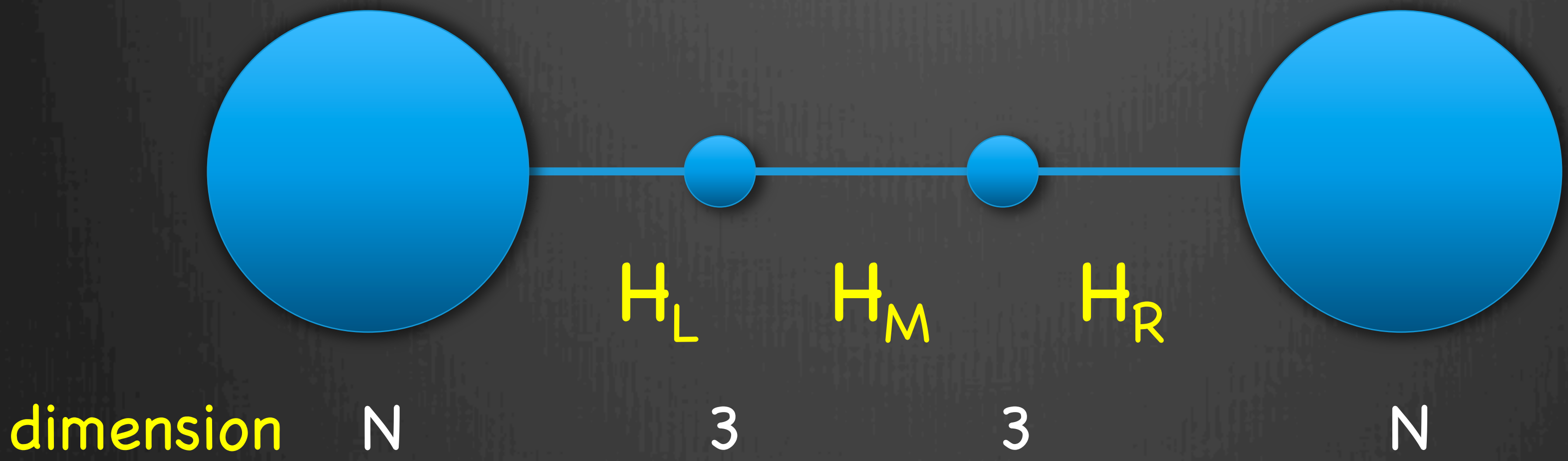
Random unitaries satisfy $\lambda \approx 1 / t^{1/2}$ [Hastings '07]

Efficient constructions (i.e. polylog(N) gates) achieve $\lambda \leq 1 / t^c$ for $c>0$. [various]

Recall communication is $\log(t) = O(\log 1/\lambda)$

Hamiltonian construction

Start with quantum expander: {I, U, V}



$$H_L = -\text{proj span}\,\{\, |\psi\rangle \otimes |1\rangle + U|\psi\rangle \otimes |2\rangle + V|\psi\rangle \otimes |3\rangle \}$$
$$H_R = -\text{proj span}\,\{\, |1\rangle \otimes |\psi\rangle + |2\rangle \otimes U^T|\psi\rangle + |3\rangle \otimes V^T|\psi\rangle \}$$
$$H_M = (|12\rangle - |21\rangle)(\langle 12| - \langle 21|) + (|13\rangle - |31\rangle)(\langle 13| - \langle 31|)$$

ground state

H_L = -proj span { $|\psi\rangle \otimes |1\rangle + U|\psi\rangle \otimes |2\rangle + V|\psi\rangle \otimes |3\rangle$}

H_R = -proj span { $|1\rangle \otimes |\psi\rangle + |2\rangle \otimes U^T|\psi\rangle + |3\rangle \otimes V^T|\psi\rangle$}

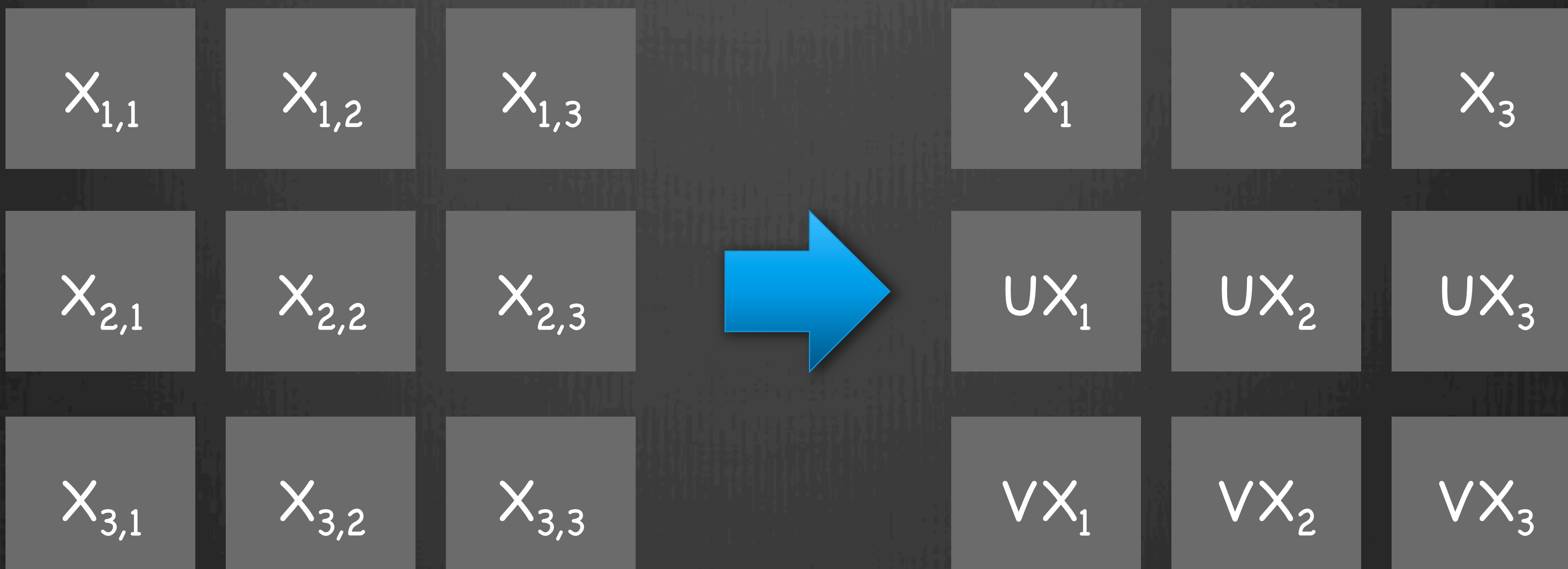
$H_M = (|12\rangle - |21\rangle)(\langle 12| - \langle 21|) + (|13\rangle - |31\rangle)(\langle 13| - \langle 31|)$

$$|\psi_0\rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} |i, j\rangle \, |X_{i,j}\rangle \cong$$

$X_{1,1}$	$X_{1,2}$	$X_{1,3}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$

constraints

$$H_L = -\text{proj span}\{\ |\psi\rangle\otimes|1\rangle + U|\psi\rangle\otimes|2\rangle + V|\psi\rangle\otimes|3\rangle\}$$



constraints

$$H_R = -\text{proj span}\{\, |1\rangle \otimes |\psi\rangle + |2\rangle \otimes U^T|\psi\rangle + |3\rangle \otimes V^T|\psi\rangle \,\}$$

X_1	X_2	X_3
UX_1	UX_2	UX_3
VX_1	VX_2	VX_3

→

X	XU	XV
UX	UXU	UXV
VX	VXU	VXV

constraints

$H_M = (|12\rangle - |21\rangle)(\langle 12| - \langle 21|) + (|13\rangle - |31\rangle)(\langle 13| - \langle 31|)$

forces $X_{1,2} = X_{2,1}$ and $X_{1,3} = X_{3,1}$

X	XU	XV
UX	UXU	UXV
VX	VXU	VXV

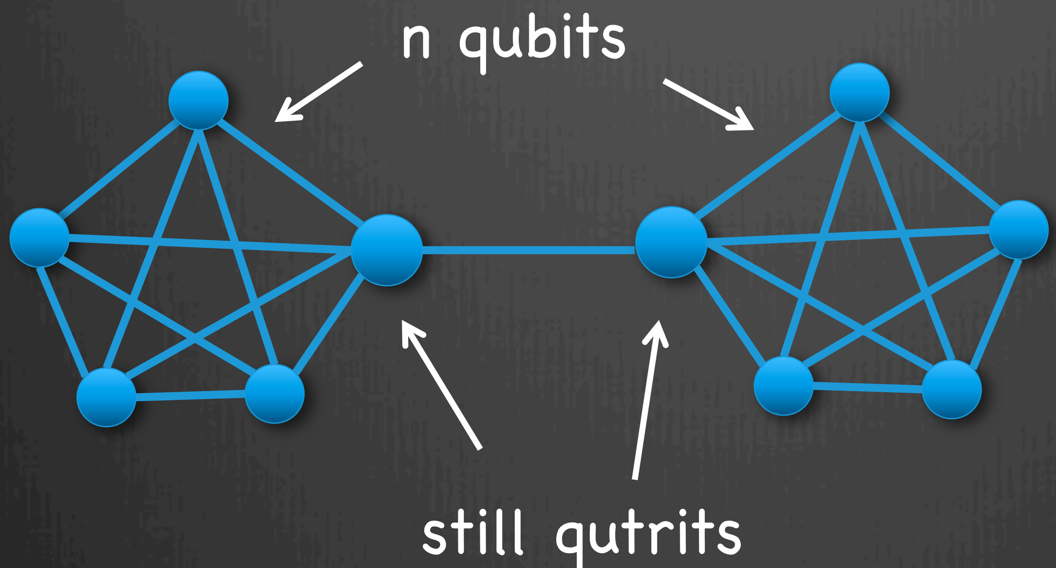
$XU = UX$

$XV = VX$

$X \propto I_N \cong |\Phi_N\rangle$

expander has constant λ → H has constant gap

qubit version



entanglement
$\propto n^c$

strategy:

1. use efficient expanders
2. use Feynman-Kitaev history state Hamiltonian
3. amplify by adding more qubits

in more detail

1. Let $N = 2^n$.
   Efficient expanders require poly(n) two-qubit gates to implement U_i, given i as input.

2. History state:

Given a circuit with gates $V_1, \ldots, V_T$

A history state is of the form

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} |t\rangle \otimes V_t V_{t-1} \cdots V_1 |\psi\rangle$$

Ground state of the Feynman-Kitaev(-Kempe-...) Hamiltonian

$$H = -\sum_t |t+1\rangle\langle t| \otimes V_t + |t\rangle\langle t+1| \otimes V_t^\dagger$$

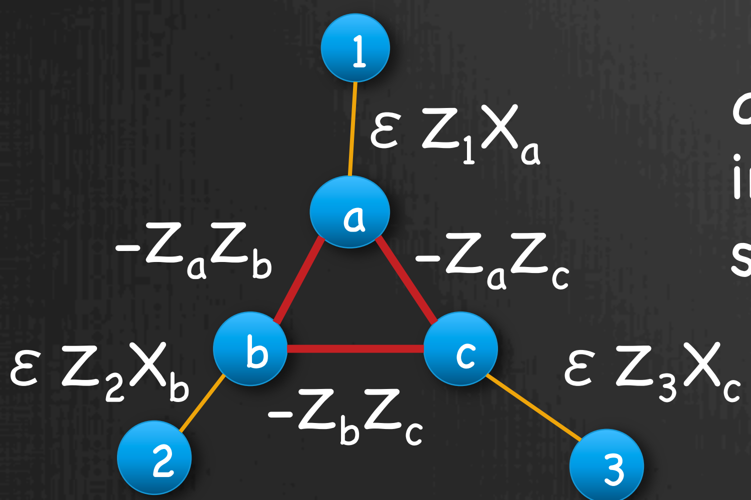
amplification

Circuit size is poly(n) → gap ∝ 1/poly(n)

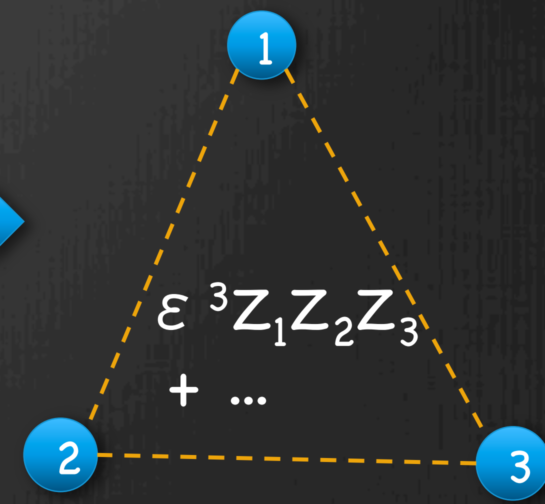
(Highly entangled ground states are known to exist in this case, even in 1-D [Gottesman-Hastings, Irani].)

idea: amplify H_L and H_R by repeating gadgets [Cao-Nagaj]

gadgets: [Kempe-Kitaev-Regev ‘03]



abc qubits ≈ in {|000⟩,|111⟩} subspace



summary

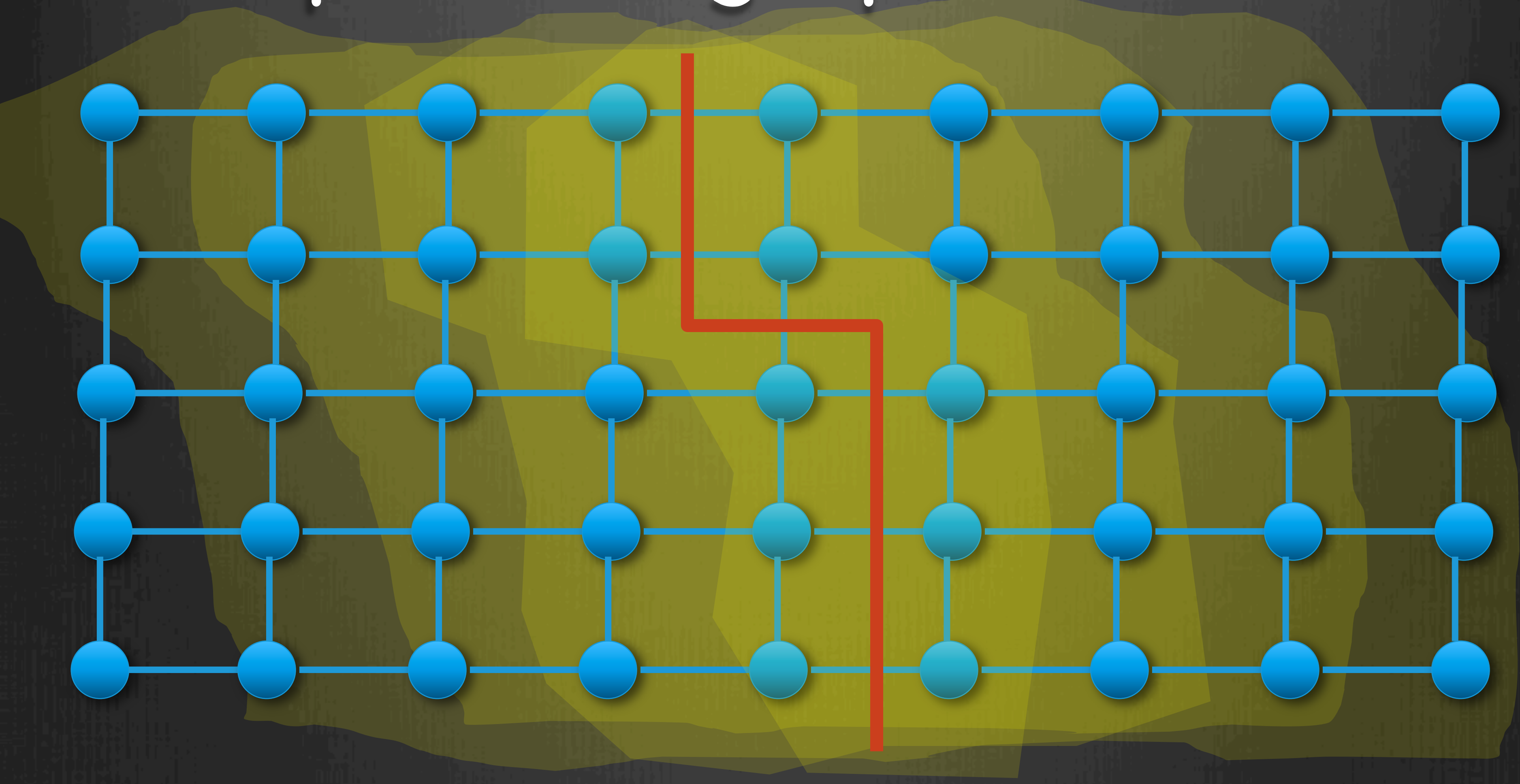
1. EPR-testing with error λ using $O(\log 1/\lambda)$ qubits

(Note: testing whether a state equals $|\psi\rangle$ requires communication $\approx \Delta(\psi)$:= "entanglement spread" of ψ.)

2. Hamiltonian with $O(1)$ gap and $n^{\Omega(1)}$ entanglement.
caveat: requires either large local dimension or large degree.

Questions:
O(1) degree, O(1) local dimension?
Qutrits in the middle → qubits?
Longer chain in the middle?
Lattices vs. general graphs

possible graph area law



conjecture: entanglement $\leq \Sigma_v \log(\dim(v)) \exp(-\mathrm{dist}(v, \mathrm{cut}) / \xi)$