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On the Stability of Asynchronous Iterative Processes*

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Abstract. We consider an iterative process in which one out of a finite set of possible operators is applied at each iteration. We obtain necessary and sufficient conditions for convergence to a common fixed point of these operators, when the order at which different operators are applied is left completely free, except for the requirement that each operator is applied infinitely many times. The theory developed is similar in spirit to Lyapunov stability theory. We also derive some very different, qualitatively, results for partially asynchronous iterative processes, that is, for the case where certain constraints are imposed on the order at which the different operators are applied.

1. Introducton

The problem investigated in this paper is the following: we are given a set $T_1, \ldots, T_K$ of operators on a common space $\mathscr{X}$ with a unique common fixed point x^*. These operators are to be applied successively, starting from an arbitrary initial element of $\mathscr{X}$. We derive necessary and sufficient conditions under which the outcome of such a sequence of operations converges to the desired common fixed point, when the order at which the operators are applied is left free; we only impose the requirement that each operator is applied an infinite number of times. (A process of this type will be called a "totally asynchronous" iterative process.)

Our main results may be expressed in the following general form: convergence is obtained if and only if there exists a Lyapunov function (suitably defined)

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which testifies to this.[1] So these results may be viewed as direct and converse Lyapunov stability theorems for a class of time-varying systems. An interesting feature is that these results are true while imposing on $\mathcal{X}$ the minimal topological structure required to define convergence to a point. The price to be paid for this level of generality is that the suitable definition of a Lyapunov function is fairly delicate and the proof of the most general converse stability theorem is based on transfinite induction.

There are several asynchronous iterative processes of practical interest [20], [22], [23] whose totally asynchronous version diverges. Nevertheless, these processes become convergent once we impose the assumption that every operator is applied at least once every M steps, where M is a suitable constant; iterative processes obeying this restriction will be called "partially asynchronous." Special cases of partially asynchronous iterative processes will also be studied in this paper, in order to contrast them to totally asynchronous processes which is our main subject.

On the practical side our results are primarily relevant to asynchronous distributed computation and chaotic relaxation algorithms [1]–[3], [8], [11], [12] and other types of asynchronous distributed systems [4], [21]. In particular, they suggest a unified methodology for analyzing and designing distributed algorithms. It is precisely this application area which has motivated the results presented in this paper. Our model of asynchronous processes is related to models of "communicating sequential processes" [10], as well as to models of "discrete event systems" [16], [17], and our results may be relevant to these contexts as well.

1.1. Related Research

The content of this paper is related to several ideas which have originated in different contexts and which we outline below.

Brayton and Tong [5], [6] have developed algorithms for deciding on the stability of nonlinear systems and time-varying systems in which the nature of the time variations is not *a priori* known. The starting point for their development is the result that the set of all products of a finite set $\{A_1, \ldots, A_K\}$ of matrices is bounded if and only if there exists a convex neighborhood V of the origin such that $A_k V \subset V$, $\forall A_k$. Our results are similar in spirit. They are more general however because nonlinear operators are allowed. Furthermore, our requirement that each operator is applied an infinite number of times significantly alters the situation.

There are some classical results in numerical analysis comparing relaxation algorithms when the variables are relaxed in different orders. For example, a slight modification of the proof of the Stein–Rosenberg theorem [24] shows that relaxation algorithms for nonnegative matrices of spectral radius less than one always converge, no matter what the order of relaxation is, provided that each

[1] Loosely speaking, a Lyapunov function for an operator T is a function $V: \mathcal{X} \to [0, \infty)$ such that $V(x^*) = 0$ and having the property $V(Tx) < V(x), \forall x \neq x^*$. Under certain conditions such a function can be used to demonstrate that the sequence $T^n x$ converges to x^*.

variable is relaxed infinitely many times. The same is true in the context of the solution of linear systems with a positive definite matrix. Such results typically amount to showing that every relaxation step decreases the value of a suitable Lyapunov-type fucntion.

Several authors have obtained sufficient conditions for convergence of asynchronous distributed or chaotic relaxation algorithms [1]-[3], [8], [11], [12], [18], [19] for linear and nonlinear problems. Chaotic relaxation differs from the relaxation algorithms mentioned in the preceding paragraph, in that a possibility of using outdated values of the variables is introduced. In fact, Chazan and Miranker obtained necessary and sufficient conditions for convergence, for the case of chaotic relaxation algorithms for the solution of linear equations [8]. There are no necessary conditions for convergence, however, for the case of nonlinear iterative processes.

1.2. Overview

In Section 2 we introduce the basic terminology and notation together with the appropriate concepts of stability and convergence of a totally asynchronous iterative process. Section 3 introduces a simple example for motivation purposes. In Section 4 we present preliminary versions of our results pertaining to the case where the space $\mathscr{X}$ is a finite set. The results of Section 4 suggest the generalizations derived in Section 5, which contains our main results. Moreover, we show, in Section 5, that our results cannot be substantially improved. In Section 6 we examine partially asynchronous processes for the case of a finite state space $\mathscr{X}$ and for the case of linear operators on a Euclidean space. The results are in sharp contrast to the results of Sections 4 and 5 and show that partially asynchronous processes are qualitatively very different from totally asynchronous ones. Section 7 contains our conclusions.

2. Problem Definition

In this section we pose the problem to be studied in the main part of this paper. We also collect here the definitions and notation to be used later.

The basic objects we will be dealing with are:

(a) A set $\mathscr{X}$ and a point $x^* \in \mathscr{X}$.

(b) A collection of functions ("operators") $T_k: \mathscr{X} \mapsto \mathscr{X}$, $k = 1, 2, \ldots, K$, satisfying $T_k x^* = x^*$, $\forall k$.

Let $\mathscr{S}$ be the set of all sequences taking values in $\{1, \ldots, K\}$. any $s \in \mathscr{S}$ will be called an *execution sequence*. We also let

$$\mathscr{S}_0 = \{s \in \mathscr{S}: s^{-1}(\{k\}) \text{ is infinite}, \forall k \in \{1, \ldots, K\}\},$$

that is, the set of all execution sequences in which each operator T_k is applied an infinite number of times. Given an initial point $x_0 \in \mathscr{X}$ and an execution sequence $s \in \mathscr{S}$, we define the corresponding *trajectory* by $x^s(0, x_0) = x_0$ and

$$x^s(n, x_0) = T_{s(n)} x^s(n-1, x_0).$$

The main question we are interested in is whether $x^s(n, x_0)$ converges to x^* (as $n \to \infty$) for all $s \in \mathcal{S}_0$ and for all $x_0 \in \mathcal{X}$. Of course, to make such a question precise we need to define a notion of convergence. Since we are interested in convergence to a single point x^*, a topological structure on the entire set $\mathcal{X}$ is not needed. We only need the following.

Definition 2.1. A collection $\mathcal{U}$ of subsets of $\mathcal{X}$ is a *neighborhood system* (around x^*) if:

(i) $x^* \in U$, $\forall U \in \mathcal{U}$.
(ii) For all $y \in \mathcal{X}$ such that $y \neq x^*$, there exists some $U \in \mathcal{U}$ such that $y \notin U$.
(iii) $\mathcal{U}$ is closed under finite intersections.
(iv) $\mathcal{U}$ is closed under unions.

Let $\mathcal{U}$ and $\mathcal{W}$ be neighborhood systems. We define some more terms:

1. We say that $\mathcal{U}$ is *finer* than $\mathcal{W}$ if for all $W \in \mathcal{W}$ there exists some $U \in \mathcal{U}$ such that $U \subset W$.[2] We also say that $\mathcal{U}$ and $\mathcal{W}$ are *equivalent* if each is finer than the other.
2. We say that $\mathcal{U}$ has a *countable base* if there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of elements of $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists some n such that $U_n \subset U$.
3. We say that a sequence $\{x_n\}_{n=0}^{\infty}$ of elements of $\mathcal{X}$ *converges* to x^* (with respect to $\mathcal{U}$) if for every $U \in \mathcal{U}$ there exists a positive integer N such that $x_n \in U$, $\forall n \geq N$.
4. A set $V \subset \mathcal{X}$ is *invariant* if $T_k V \subset V$, $\forall k$. Finally, a neighborhood system consisting exclusively of invariant sets is called an *invariant neighborhood system.*

We continue with a few remarks.

1. If $\mathcal{U}$ is a neighborhood system, then $\bigcap_{U \in \mathcal{U}} U = \{x^*\}$. Nevertheless, $\{x^*\} \notin \mathcal{U}$, in general.
2. If $\mathcal{X}$ is endowed with a topology which separates points,[3] a natural neighborhood system is given by $\mathcal{U} = \{U: U \text{ is open and contains } x^*\}$.
3. Our development and the results of Sections 4–6 generalize to the case where x^* is a subset of $\mathcal{X}$ rather than a single point. The obvious modifications in the definition of a neighborhood system are: (i) $x^* \subset U$, $\forall U \in \mathcal{U}$, and (ii) for every $y \notin x^*$ there exists some $U \in \mathcal{U}$ such that $y \notin U$.
4. Finer neighborhood systems correspond to stronger requirements for convergence: if $\mathcal{U}$ is finer than $\mathcal{W}$ and $\{x_n\}$ converges with respect to $\mathcal{U}$, it also converges with respect to $\mathcal{W}$. The converse is generally false.

We conclude by defining the concepts of stability and convergence to be employed.

[2] The symbol $\subset$ denotes set inclusion, not necessarily proper.

[3] A topology "separates points" if for every $x, y \in \mathcal{X}$ there exists an open set U such that $x \in U$ and $y \notin U$.

Definition 2.2. Given $\mathscr{X}$, $\mathscr{U}$, x^*, T_k, $k = 1, 2, \ldots, K$, we say that $\mathscr{S}_0$ is *stable* if

$$\forall U \in \mathscr{U}, \exists V \in \mathscr{U} \quad \text{such that} \quad \forall x_0 \in V, \forall s \in \mathscr{S}_0, \forall n, \qquad x^s(n, x_0) \in U.$$

We also say that $\mathscr{S}_0$ *converges* if $x^s(n, x_0)$ converges to x^* for all $x_0 \in \mathscr{X}$ and for all $s \in \mathscr{S}_0$.

Notice that our definition of stability is similar to the usual concept of stability of dynamical systems, whereas our notion of convergence corresponds to the usual concept of asymptotic stability. Let us point out here that neither of the two concepts defined above implies the other, in general.

3. An Example

A simple example, to illustrate our model, is the distributed gradient algorithm [3], [20], [23] for minimizing a function $f: R^n \mapsto R$. This algorithm operates as follows: to each component of the vector with respect to which we are optimizing we associate a particular processor i ($i \in \{1, \ldots, n\}$). Each processor i keeps in its memory a vector $x^i \in R^n$ and once in a while updates the ith component of x^i according to

$$x_i^i \leftarrow x_i^i - \gamma \frac{\partial f}{\partial x_i}(x^i),$$

where $\gamma > 0$ is a (typically small) stepsize. For any $j \neq i$, processor i also receives once in a while messages from processor j containing the value x_j^j of the jth component, as computed by processor j; upon receipt of such a message, processor i updates its own jth component according to $x_j^i \leftarrow x_j^j$. We assume that the communication delays are zero. We are interested in the question whether such an algorithm converges to a stationary point of f without imposing any timing assumptions on the sequence of computations and communications by each processor, other than a requirement that no processor ever quits. Several sufficient conditions for the convergence of this algorithm are known [1], [3], [20], [23].

In order to recast the above algorithm into our framework we identify $\mathscr{X}$ with R^{n^2}. In particular, we define a vector $x = (x^1, x^2, \ldots, x^n)$ consisting of n subvectors, each of dimension n. Such a vector provides a complete description of the state of all processors at any given time. This vector is modified by a communication or a computation by some processor. Both of these ways of modifying the vector x may be viewed as special kinds of operators $T: \mathscr{X} \mapsto \mathscr{X}$. (So, in this formulation, the distinction between communications and computations is ignored.) Moreover, the assumption that no processor ever quits corresponds to the assumption that each operator is applied an infinite number of times.

If, say bounded, communication delays are allowed, then we may still recast the algorithm into our framework using the standard method of state augmentation. With unbounded communication delays, however, a different approach may be required.

Typical proofs of convergence [3] of the distributed gradient algorithm amount to constructing an appropriate nested sequence of subsets of R^n, whose intersection is the minimizing point of f, and showing that once the state enters such a set it never leaves it and eventually moves into the next smaller set. In the next sections we essentiallly investigate the extent to which this technique is a generic method, universally applicable to asynchronous iterative processes.

4. The Finite Case

Substantial insights may be obtained by looking first at the special case where $\mathscr{X}$ is a finite set, which we will assume in this section. Some of the results presented in this section are very easy to obtain; their merit, however, is that they suggest the appropriate generalizations to the case where $\mathscr{X}$ is infinite.

When $\mathscr{X}$ is finite, any neighborhood system $\mathscr{U}$ must contain the singleton $\{x^*\}$. It follows that a sequence x_n converges to x^* if and only if $x_n = x^*$, for all n larger than some finite n_0. Let us also point out that the issue of stability is trivial because $\mathscr{S}_0$ is always stable. (Simply let the set v in the definition of stability, in Section 2, to be equal to $\{x^*\}$.)

With $\mathscr{X}$ finite, an asynchronous iterative process may be conveniently described by means of a "colored directed graph," each color corresponding to a transition resulting from the application of a different operator. This graph is constructed as follows: let $V = \mathscr{X}$ be the set of nodes of the graph. A *colored edge* is an ordered triple (i, j, k) such that $i \in V$, $j \in V$, and $k \in \{1, \ldots, K\}$. We say that k is the color of the directed edge (i, j). Given the set $\{T_1, \ldots, T_K\}$ of operators, we introduce the following set E of colored edges: $E = \{(i, j, k): T_k(i) = j\}$. A *walk* is a finite sequence of colored edges $\{(i_1, j_1, k_1), \ldots, (i_n, j_n, k_n)\}$ such that (i) $(i_m, j_m, k_m) \in E$, $\forall m \in \{1, \ldots, n\}$, and (ii) $j_m = i_{m+1}$, $m \in \{1, 2, \ldots, n-1\}$. A *cycle* is a walk satisfying $i_1 = j_n$.

Theorem 4.1. *Assuming that $\mathscr{X}$ is finite, the following are equivalent.*

(a) *$\mathscr{S}_0$ converges.*

(b) *There exists no cycle which uses all colors but does not go through the node x^*.*

(c) *There exists a finite, totally ordered index set A and a collection $\{X_\alpha: \alpha \in A\}$ of subsets of $\mathscr{X}$ with the following properties:*

(i) $\alpha < \beta \Rightarrow X_\alpha \subset X_\beta$.

(ii) $\bigcap_{\alpha \in A} X_\alpha = \{x^*\}$.

(iii) $\bigcup_{\alpha \in A} X_\alpha = \mathscr{X}$.

(iv) $T_k X_\alpha \subset X_\alpha$, $\forall k, \alpha$.

(v) *For any $\alpha \in A$ such that $X_\alpha \neq \{x^*\}$, there exists some $i(\alpha) \in \{1, \ldots, K\}$ and some $\beta < \alpha$ such that $T_{i(\alpha)} X_\alpha \subset X_\beta$.*

Proof. (a)$\Rightarrow$(b) If there was a cycle using all colors, then some execution sequence in $\mathscr{S}_0$ could traverse this walk an infinite number of times without ever converging to x^*.

(b)⇒(a) If there is no such cycle then, for any $s \in \mathcal{S}_0$ and any $x_0 \neq x^*$, the resulting trajectory may visit the point x_0 only a finite number of times, that is, until all colors have been used at least once. By the same argument, any other point on this trajectory is visited only a finite number of times. Since there are only finitely many points, x^s must settle to x^* in finite time.

(c)⇒(a) This is trivial.

(a)⇒(c) The proof is omitted because this is an easy corollary of the more general Theorem 5.2. Let us simply state here that the sets X_α may be constructed as follows. Let A be a finite subset $\{0, \ldots, N\}$ of the integers, with N large enough. Let $X_0 = \{x^*\}$. Having defined X_α, let $X_{\alpha+1}$ be a minimal invariant subset of $\mathcal{X}$ properly containing X_α. It is left to the reader to verify that these sets have all the desired properties. □

Condition (b) above is more natural than condition (c). However, condition (b) cannot be generalized to the case of infinite sets: a sequence may be nonconvergent without ever taking the same value twice.

Given a colored graph, there are very simple polynomial time algorithms for deciding whether condition (b) holds. (We only need to examine whether there exists some $x \in \mathcal{X}$, different from x^*, such that, for each color k, there exists a cycle through x which uses color k.) Certain asynchronous algorithms which are of interest in computer science [9] are exactly of the type considered in this section and one may want to have an automatic procedure for proving that such algorithms operate correctly. (For example, it is mentioned in [15] that the correctness of an algorithm for the critical section problem was first proved automatically by a computer.) Condition (b) together with a polynomial time algorithm for testing it may be viewed as an efficient automatic proof procedure.

We close this section by indicating the connection between condition (c) and Lyapunov stability theory. We may define a function $V: \mathcal{X} \mapsto A$ by $V(x) = \min\{\alpha \in A: x \in X_\alpha\}$. Then, V essentially plays the role of a Lyapunov function for the iterative process under consideration: its value never increases and occasionally it has to decrease. In our formulation of the results, however, we have preferred to work with the level sets of the functioion V. This will be done in the next section as well, but it should be kept in mind that all results have simple counterparts in terms of Lyapunov functions.

5. The Infinite Case

We first settle the issue of stability. The following result effectively generalizes the result of Brayton and Tong [5], [6] mentioned in Section 1 which pertained to the case of matrices operating on Euclidean spaces.

Theorem 5.1. *The following are equivalent*:

(a) S_0 *is stable* (*with respect to* $\mathcal{U}$).

(b) *There exists an invariant neighborhood system* $\mathcal{W}$ *which is equivalent to* $\mathcal{U}$.

Proof. (a)⇒(b) We assume that $\mathcal{S}_0$ is stable and we have to construct a collection $\mathcal{W}$ of subsets of $\mathcal{X}$ with the desired properties. We do this as follows. Given any

$U \in \mathcal{U}$ we define W_U as the union of all invariant subsets of U. Notice that $\{x^*\}$ is an invariant subset of U, which shows that W_U is nonempty, $\forall U \in \mathcal{U}$. Moreover, notice that W_U is the largest invariant subset of U. Let $\mathcal{W}' = \{W_U : U \in \mathcal{U}\}$ and let $\mathcal{W}$ be the closure of $\mathcal{W}'$ under finite intersections and arbitrary unions. We will show that $\mathcal{W}$, so constructed, has all the desired properties.

Referring to the definition of a neighborhood system (Definition 2.1) we see that (i) holds because $x^* \in W_U$, $\forall U \in \mathcal{U}$. Properties (iii) and (iv) hold by construction. Finally, for property (ii), notice that for any $y \neq x^*$ there exists some $U \in \mathcal{U}$ such that $y \notin U$; it follows that $y \notin W_U$. Since W_U belongs to $\mathcal{W}$, we conclude that $\mathcal{W}$ is indeed a neighborhood system. Given that the intersection or the union of invariant subsets of $\mathcal{X}$ is invariant, it follows that $\mathcal{W}$ is in fact an invariant neighborhood system.

It remains to show that $\mathcal{W}$ is equivalent to $\mathcal{U}$. By construction, $W_U \subset U$, $\forall U \in \mathcal{U}$. Therefore, $\mathcal{W}$ is finer than $\mathcal{U}$. In order to show that $\mathcal{U}$ is finer than $\mathcal{W}$ it is sufficient[4] to show that for every $W_U \in \mathcal{W}'$ there exists some $V \in \mathcal{U}$ such that $V \subset W_U$. (This is because $\mathcal{W}'$ "generates" $\mathcal{W}$.)

Let $U \in \mathcal{U}$. Using the stability assumption, there exists some $V \in \mathcal{U}$ such that $V \subset U$ and such that any trajectory starting in V stays inside U. Let V' be the set of all points lying on any trajectory which starts in V. Clearly, $V \subset V' \subset U$ and V' is invariant. Since W_U is the largest invariant subset of U, we have $V \subset V' \subset W_U$. This completes this direction of the proof.

(b)$\Rightarrow$(a) Given any $U \in \mathcal{U}$, there exists some $W \in \mathcal{W}$ such that $W \subset U$, because $\mathcal{W}$ is finer than $\mathcal{U}$. Moreover, since $\mathcal{U}$ is finer than $\mathcal{W}$, there exists some $V \in \mathcal{U}$ such that $V \subset W$. Any trajectory which starts in V must remain inside W, because $V \subset W$ and W is invariant. Since $W \subset U$, it follows that any such trajectory has to remain inside U as well, which shows that $\mathcal{S}_0$ is stable and completes the proof. □

We now turn to the question of convergence of $\mathcal{S}_0$. We introduce the following conditon which generalizes condition (c) of Theorem 4.1.

Condition 5.1. *There exists a totally ordered index set A and a collection $\{X_\alpha : \alpha \in A\}$ of distinct nonempty subsets of $\mathcal{X}$ with the following properties:*

(i) $\alpha < \beta \Rightarrow X_\alpha \subset X_\beta$.
(ii) *For every $U \in \mathcal{U}$ there exists some $\alpha \in A$ such that $X_\alpha \subset U$.*
(iii) $\bigcup_{\alpha \in A} X_\alpha = \mathcal{X}$.
(iv) $T_k X_\alpha \subset X_\alpha$ *for all k and all $\alpha \in A$.*
(v) *For every $\alpha \in A$ such that $X_\alpha \neq \{x^*\}$ there exists some $i(\alpha) \in \{1, \ldots, K\}$ such that $T_{i(\alpha)} X_\alpha \subset \bigcup_{\beta < \alpha} X_\beta$.*
(vi) *Every nonempty subset of A which is bounded below has a smallest element.*

Theorem 5.2.

(a) *If Condition 5.1 holds, then $\mathcal{S}_0$ converges.*
(b) *If $\mathcal{S}_0$ is stable and converges and if $\mathcal{U}$ has a countable base, then Condition 5.1 holds.*

[4] This is not entirely obvious but is a standard technique in point set topology.

Proof. (a) Let $A, \{X_\alpha : \alpha \in A\}$ have the properties in Condition 5.1. Suppose that we are given some $U \in \mathcal{U}$, $x_0 \in \mathcal{X}$, $s \in \mathcal{S}_0$. We must show that $x^s(n, x_0)$ eventually enters and remains in U. Let

$$B = \{\alpha \in A : \exists n \text{ such that } x^s(n, x_0) \in X_\alpha\}.$$

Lemma 5.1. $B = A$.

Proof of Lemma 5.1. Since $\mathcal{X} = \bigcup_{\alpha \in A} X_\alpha$, there exists some $\alpha \in A$ such that $x^s(0, x_0) \in X_\alpha$. Hence B is nonempty. We consider two cases: we first assume that B is not bounded below. Then, for every $\alpha \in A$, there exists some $\beta \in B$ such that $\beta < \alpha$. Hence for every $a \in A$ there exists some $\beta < \alpha$ and some integer n such that $x^s(n, x_0) \in X_\beta \subset X_\alpha$. So, every $\alpha \in A$ belongs to B and $A = B$.

Let us now assume that B is bounded below. Since it is nonempty, it has a smallest element (Condition 5.1(vi)), denoted by β. If $X_\beta = \{x^*\}$, then β is also the smallest element of a and $A = B$ follows. So, let us assume that $X_\beta \neq \{x^*\}$. From the definition of B there exists some n_0 such that $x^s(n_0, x_0) \in X_\beta$ and (by invariance of X_β) $x^s(n, x_0) \in X_\beta$ for all $n \geq n_0$. Since $s \in \mathcal{S}_0$, there exists some $m > n_0$ such that $s(m) = i(\beta)$, where $i(\beta)$ has been defined in Condition 5.1(v). Therefore, there exists some $\gamma < \beta$ such that $x^s(m, x_0) = T_{i(\beta)} x^s(m-1, x_0) \in X_\gamma$. Hence $\gamma \in B$ which contradicts the definition of β as the smallest element of B. This completes the proof of the lemma. □

Given $U \in \mathcal{U}$, there exists some $\alpha \in A$ such that $X_\alpha \subset U$ (Condition 5.1(ii)) and, since $B = A$, there exists some n_0 such that $x^s(n_0, x_0) \in X_\alpha$. By the invariance of X_α, $x^s(n, x_0) \in X_\alpha \subset U$ for all $n \geq n_0$, which completes this direction of the proof.

(b) We assume that $\mathcal{U}$ has a countable base and that $\mathcal{S}_0$ is stable and converges (with respect to $\mathcal{U}$). Using Theorem 5.1, there exists another neighborhood system $\mathcal{W}$ consisting of invariant subsets only and which is equivalent to $\mathcal{U}$. Since $\mathcal{U}$ has a countable base, it is easy to see that $\mathcal{W}$ has a countable base as well. This shows that, without any loss of generality, we may assume that $\mathcal{U}$ consists of invariant sets only.

Let $\{U_n\}_{n=1}^{\infty}$ be a countable base of $\mathcal{U}$ and let $U_0 = \mathcal{X}$. Without any loss of generality we may assume that $U_{n+1} \subset U_n$, $\forall n$. (Otherwise we could define a new countable base by $U'_n = \bigcap_{k=0}^{n} U_k$.)

Our proof consists of two main steps: for each n we construct a nested collection of subsets of $\mathcal{X}$ which lie between U_n and U_{n+1}; then we merge these collections to get a single nested collection.

Lemma 5.2. *Let V be an invariant subset of $\mathcal{X}$ and let I be the set of all invariant subsets of V. Then there exist functions $p: I \mapsto I$ and $i: I \mapsto \{1, \ldots, K\}$ such that:*

(i) *For any $U \in I$ we have $p(U) \supset U$ and if $U \neq V$, then $p(U) \neq U$.*
(ii) $T_{i(p(U))} p(U) \subset U$ *for all* $U \in I$.

Proof of Lemma 5.2. For any $x \in \mathscr{X}$, let $R(x)$ be the set of all points belonging to some trajectory with initial point x. Given some $U \in I$ which is not equal to V and any $j \in \{1, \ldots, K\}$ let

$$p_j(U) = \{x: T_j y \in U, \forall y \in R(x)\}.$$

Clearly, $p_j(U)$ is invariant, $p_j(U) \supset U$, and $T_j p_j(U) \subset U$. We now show that there exists some j for which the inclusion $p_j(U) \supset U$ is proper. Suppose the contrary. Then, for every $x \in V$ such that $x \notin U$, we have $x \notin p_j(U)$, $\forall j$. That is, for any such x there exists a trajectory which leads to some point y for which $T_j y \notin U$. Since this is true for each j we can piece together such trajectories to obtain an infinite trajectory in which all the T_k's are applied an infinite number of times but which never enters the set U. This contradicts the assumption that $\mathscr{S}_0$ converges. We can now define $p(U)$ to be equal to $p_j(U)$ for some j for which the inclusion $p_j(U) \supset U$ is proper and let $i(p(U))$ be equal to that particular j, which completes the proof of the lemma. □

Let A^n be a well-ordered set with cardinality larger than that of $\mathscr{X}$ and let α_0^n be its smallest element.[5] We apply Lemma 5.2, with $V = U_n$, to obtain a function p_n, satisfying properties (i) and (ii) of that lemma. We then define a function $h^n: A^n \mapsto I$ using the following tansfinite recursion:

$$h^n(\alpha_0^n) = U_{n+1}$$

and, for $\alpha > \alpha_0^n$,

$$h^n(\alpha) = p_n\left(\bigcup_{\beta<\alpha} h^n(\beta)\right).$$

Notice that $U_{n+1} \subset h^n(\beta) \subset h_n(\alpha) \subset U_n$, for any α, β such that $\alpha > \beta$, and that, if $h^n(\beta) \neq U_n$, then the containment $h^n(\alpha) \supset h^n(\beta)$ is proper. Since A^n has cardinality larger than that of $\mathscr{X}$, there exists some $\alpha \in A^n$ such that $h^n(\alpha) = U_n$. Let $\bar{\alpha}^n$ be the smallest such α and let $\bar{A}^n = \{\alpha \in A^n: \alpha < \bar{\alpha}^n\}$. Finally, for any $\alpha \in \bar{A}^n$, let $X_{(n,\alpha)} = h^n(\alpha)$.

We now carry out the last step of the proof. Having defined $\bar{A}^n$ for each n, we let $A = \{(n, \alpha): \alpha \in \bar{A}^n, n = 0, 1, 2, \ldots\}$ with the following total order: $(n, \alpha) < (m, \beta)$ if and only if $n > m$ or if $n = m$ and $\alpha < \beta$. We claim that the collection $\{X_{(n,\alpha)}: (n, \alpha) \in A\}$ has all the desired properties. Indeed, properties (i)-(iv) are true by construction. Property (v) is true because of the way that the functions h^n were defined. Finally, any nonempty subset of A which is bounded below is isomorphic to a finite union of well-ordered sets, in lexicographic order. Therefore it is itself well ordered and has a smallest element, which demonstrates that property (vi) is also true. □

[5] A set is called *well ordered* if it is totally ordered and every nonempty subset has a smallest element. See [13] for the basic properties of well-ordered sets.

Remark 1. There do not seem to be any practically interesting situations in which one has convergence but not stability, neither are there situations in which $\mathcal{U}$ does not have a countable base. With these remarks in mind, Theorem 5.2 may be reformulated as follows: suppose that $\mathcal{U}$ has a countable base and that $\mathcal{S}_0$ is stable. Then, $\mathcal{S}_0$ converges if and only if Condition 5.1 is true.

Remark 2. Parts (i)-(iv) of Condition 5.1 are the straightforward counterparts of parts (i)-(iv) of condition (c) in Theorem 4.1. Parts (v) and (vi) are more delicate. If part (v) was modified to the requirement that for every $\alpha \in A$ there exists some $i(\alpha)$ and some $\beta < \alpha$ such that $T_{i(\alpha)}X_\alpha \subset X_\beta$, then part (b) of Theorem 5.2 would be false, as may be demonstrated by simple examples. Part (vi) is also needed because without it part (a) of Theorem 5.2 would be false.

Remark 3. One might be tempted to conjecture that Condition 5.1 implies not only convergence but stability as well. This is false, as may be demonstrated by smple examples. Similarly, in part (b) of Theorem 5.2, the assumption that $\mathcal{S}_0$ is stable cannot be dispensed with. Indeed, there are examples of processes which are convergent (but not stable) and for which Condition 5.1 fails to hold.

Remark 4. We do not know whether the requirement in part (b) of the theorem, that $\mathcal{U}$ has a countable base, can be relaxed, but we conjecture it cannot. Nevertheless, it can be shown that if Condition 5.1 holds, then there exists a new neighborhood system $\mathcal{W}$ which is finer than $\mathcal{U}$, has a countable base and such that $\mathcal{S}_0$ is stable and convergent with respect to $\mathcal{W}$. In some sense, this states that for Condition 5.1 to hold, $\mathcal{U}$ can fail to have a countable base only if it has been chosen unnaturally small.

Remark 5. Nothing has been said about the cardinality of the index set A. One may ask whether the integers are always an adequate index set. We will show (Theorem 5.3) that this is not the case. In fact, we show that even more general countable index sets are not sufficient.

We now turn to the issue of the cardinality of A. We introduce some terminology which will be needed later: an *isomorphism* of well-ordered sets is a bijection which preserves the respective total orders. A *section* of a well-ordered set A is a subset of A of the form $S_\alpha = \{\beta \in A: \beta < \alpha\}$, where α is some element of A. If A and B are well ordered, we say that A is *smaller* than B (denoted by $A \prec B$) if A is isomorphic to a section of B. It is known that, for any two well-ordered sets, either one is smaller than the other or they are isomorphic [13]. We also use the notation $A \preceq B$ to indicate that either $A \prec B$ or A is isomorphic to B.

Theorem 5.3. *There exists an asynchronous iterative process (with $K = 3$) which is stable and convergent (with respect to a particular neighborhood system) and such that any collection $\{X_\alpha\}$ of subsets of $\mathcal{X}$ which has properties* (i)-(vi) *of Condition* 5.1 *is uncountable.*

Proof. Let B be an arbitrary countable well-ordered set and let β_0 be its smallest

element. Let Z be the set of integers and let $\mathscr{X} = \{(\beta, n): \beta \in B, n \in Z\} \cup \{x^*\}$. For every $\beta \in B$, $\beta > \beta_0$, let f_β be a surjective mapping from Z onto $\{\beta' \in B: \beta' < \beta\}$. We define three operators T_1, T_2, $T_3: \mathscr{X} \mapsto \mathscr{X}$ as follows. We let $T_i(x^*) = T_i(\beta_0, n) = x^*$, $\forall n \in Z$, $\forall i \in \{1, 2, 3\}$. Also, for $\beta \in B$, $\beta > \beta_0$, we let $T_1(\beta, n) = (\beta, n+1)$, $T_2(\beta, n) = (\beta, n-1)$, and $T_3(\beta, n) = (f_\beta(n), 0)$. We also let $\mathscr{U} = \{\{x^*\}\}$. It is easy to see that $\mathscr{S}_0$ is stable and convergent (with respect to $\mathscr{U}$). Indeed, if we let, for $\beta \in B$, $X_\beta = \{(\beta', n): \beta' \leq \beta\}$ and $X_0 = \{x^*\}$, then the collection $\{X_0\} \cup \{X_\beta: \beta \in B\}$ testifies to this. (In reference to condition (v) of Condition (5.1), notice that we have $i(\beta) = 3$, $\forall \beta \in B$.)

Let $\mathscr{Y} = \{Y_\alpha: \alpha \in A\}$ be a collection of invariant subsets of $\mathscr{X}$ which satisfies Condition 5.1. Then $\{x^*\}$ belongs to $\mathscr{Y}$ and is its smallest element. Hence A is bounded below and is therefore well ordered. We will show that $B \preceq A$. This is obvious if $\mathscr{Y} = \{X_\beta: \beta \in B \cup \{0\}\}$, so suppose that this is not the case. Let β_1 be the smallest $\beta \in B$ such that $X_\beta \notin \mathscr{Y}$ and let β_2 be the smallest $\beta \in B$ such that $\beta > \beta_1$. Let Y_α be the smallest element of $\mathscr{Y}$ containing $(\beta_2, 0)$. Since Y_α is invariant under T_i, $i = 1, 2, 3$, we have $Y_\alpha \supset X_{\beta_2}$. By the definition of Y_α, we have $(\beta_2, 0) \notin Y_\gamma$, $\forall \gamma < \alpha$. Also, $(\beta_1, 0) \notin Y_\gamma$, $\forall \gamma < \alpha$, because otherwise we would have (by the invariance of Y_γ) $(\beta_1, n) \in Y_\gamma$, $\forall n \in Z$, and $Y_\gamma = X_{\beta_1}$, which would contradict the definition of β_1. By part (v) of Condition 5.1 there exists some $i(\alpha) \in \{1, 2, 3\}$ such that $T_{i(\alpha)} Y_\alpha \subset \bigcup_{\gamma < \alpha} Y_\gamma$. Therefore, $T_{i(\alpha)} x \neq (\beta_1, 0)$ and $T_{i(\alpha)} x \neq (\beta_2, 0)$, $\forall x \in Y_\alpha$. If $i(\alpha) = 1$, then $(\beta_2, -1) \in Y_\alpha$ and $T_1(\beta_2, -1) = (\beta_2, 0)$. So, $i(\alpha) \neq 1$ and the same argument shows that $i(\alpha) \neq 2$. Finally, if $i(\alpha) = 3$, let $n \in Z$ be such that $f_{\beta_2}(n) = \beta_1$. Then, $T_3(\beta_2, n) = (f_{\beta_2}(n), 0) = (\beta_1, 0)$ which is a contradiction. We may therefore conclude that $B \preceq A$.

Having shown that arbitrarily "large" countable well-ordered index sets may be required, we construct, by diagonalization, an example in which an uncountable index set is needed.

Let Ω be the smallest uncountable well-ordered set [13] and consider its sections $S_\omega = \{\omega' \in \Omega: \omega' < \omega\}$. It is known that each S_ω is countable. Using our previous construction, there exists, for each $\omega \in \Omega$, a stable and convergent asynchronous iterative process $P_\omega = (\mathscr{X}_\omega, x^*_\omega, T^\omega_1, T^\omega_2, T^\omega_3, \mathscr{U}_\omega)$ with the property that if an index set A is sufficient to demonstrate stability and convergence, then $S_\omega \preceq A$. Moreover, our earlier construction shows that we may assume that $\mathscr{U}_\omega = \{\{x^*_\omega\}\}$. Let us identify the elements x^*_ω with each other (so, the subscript ω may be dropped) but assume that all other elements of the $\mathscr{X}_\omega$'s are distinct. Let $\mathscr{X} = \bigcup_{\omega \in \Omega} \mathscr{X}_\omega$ and define $T_k: \mathscr{X} \mapsto \mathscr{X}$, for $k = 1, 2, 3$, by $T_k x = T^\omega_k x$, if $x \in \mathscr{X}_\omega$. Finally, let $\mathscr{U} = \{\{x^*\}\}$. The process $P = (\mathscr{X}, x^*, T_1, T_2, T_3, \mathscr{U})$ is stable and convergent. Moreover, since each P_ω is "imbedded" in P, if a collection $\{X_\alpha: \alpha \in A\}$ satisfies Condition 5.1, then $S_\omega \preceq A$, $\forall \omega \in \Omega$. Suppose that such a set A is countable. Since A is well ordered (because it has the smallest element $\{x^*\}$) it is isomorphic to S_{ω^*}, for some $\omega^* \in \Omega$. Therefore, $S_\omega \preceq S_{\omega^*}$, $\forall \omega \in \Omega$. This implies that $\Omega = \{\omega^*\} \cup S_{\omega^*}$. But this would imply that Ω is countable, which contradicts the definition of Ω and completes the proof. □

The above counterexample is reminiscent of stability theory for general (continuous time) systems in which Lyapunov functions taking values in sets

with cardinality larger than that of the continuum are required [7]. Notice that we have only shown that uncountable index sets are generally required. We do not know, however, whether arbitrarily larger cardinalities are required or not.

We end this section with a simpler version of Theorem 5.2 which relates to a stronger notion of convergence. As far as all conceivable practical applications are concerned, the following result seems to be adequate. The direction (b)⇒(a) below has been obtained earlier in [19]. This direction also contains the essence of the argument in [1] and [3]. We introduce some terminology. Given some $s \in \mathcal{S}_0$, we say that the set $[p, q] = \{p, p+1, \ldots, q\}$ is a cycle if for any $k \in \{1, \ldots, K\}$ there exists some integer $p_k \in [p, q]$ such that $s(p_k) = k$.

Theorem 5.4. *The following are equivalent:*

(a) *For every $U \in \mathcal{U}$ there exists some N such that $x^s(n, x_0) \in U$, for every $s \in \mathcal{S}_0$, every $x_0 \in \mathcal{X}$, and every n, such that $\{1, \ldots, n\}$ is the union of N disjoint cycles.*

(b) *There exists a family $\{X_n : n \in \mathcal{N}\}$ of subsets of $\mathcal{X}$ such that:*

(i) $X_0 = \mathcal{X}$.

(ii) $X_{n+1} \subset X_n$.

(iii) *For every $U \in \mathcal{U}$ there exists some n such that $X_n \subset U$.*

(iv) $T_i X_n \subset X_n$, $\forall i, n$.

(v) *For every n there exists some $i(n)$ such that $T_{i(n)} X_n \subset X_{n+1}$.*

Proof. (b)⇒(a) It is easy to see that, for any initial point x_0, the trajectory moves from X_n to X_{n+1} after each consecutive cycle and stays in X_{n+1} thereafter. (This is because of conditions (iv) and (v) of part (b).) Hence after N cycles, $x^s(n, x_0)$ belongs to X_N. The result follows by using the assumption that for every $U \in \mathcal{U}$ there exists some N such that $X_N \subset U$, which completes the proof of this direction of the theorem.

(a)⇒(b) For any set $Y \subset \mathcal{X}$ let $R(Y)$ denote the set of all points on any trajectory starting from a point in Y. (So, $R(Y)$ is the smallest invariant set containing Y.) Let $X_0 = \mathcal{X}$ and define X_n recursively by $X_{n+1} = R(T_{i(n)} X_n)$, where $i(n) = n - 1 \pmod K + 1$. We will show that $\{X_n\}_{n=1}^{\infty}$ has the desired properties. Properties (i), (ii), and (iv) are immediate. Property (v) also holds because $T_{i(n)} X_n \subset R(T_{i(n)} X_n) = X_{n+1}$. Finally, given any $U \in \mathcal{U}$, let N be as prescribed in statement (a) of the theorem. Notice that (by construction), for every $x \in X_{KN}$, there exists some $x_0 \in \mathcal{X}$ and some $s \in \mathcal{S}_0$ and some n such that $\{1, \ldots, n\}$ contains N disjoint cycles and such that $x^s(n, x_0) = x$. It follows, using the assumption that (a) holds, that $x \in U$. Hence $X_{KN} \subset U$ which completes the proof. □

6. Partially Asynchronous Processes

In this section we consider partially asynchronous iterative processes and contrast them to the totally asynchronous processes considered thus far. Partially asynchronous processes are of the same type as the processes introduced in Section 2. The only difference is that instead of examining the convergence of trajectories

x^s corresponding to arbitrary execution sequences s in $\mathcal{S}_0$, we are only interested in those execution sequences for which every operator T_k is applied at least once every M steps, where M is a fixed positive integer. To be more precise, we define $\mathcal{S}_M$ as the set of all $s \in \mathcal{S}_0$ such that $\{s(n+1), \ldots, s(n+M)\} = \{1, \ldots, K\}$, $\forall n \geq 0$.

Our first result shows that deciding on the convergence of a partially asynchronous process on a finite state space is, in general, a hard combinatorial problem. (For definitions and basic methods on the complexity of combinatorial problems refer to [9] and [14].)

Theorem 6.1. *Assume that $\mathcal{X}$ is a finite set of cardinality N. Given $(\mathcal{X}, x^*, T_1, \ldots, T_K, M)$, the problem of deciding whether $x^s(n, x_0)$ converges to x^* for all $s \in \mathcal{S}_M$ is NP-hard (that is, it cannot be solved in polynomial time unless $P = NP$). However, the problem is polynomial if we restrict to instances in which either $M \geq 2NK$ or to instances in which K is held constant.*

Proof. To demonstrate NP-hardness, we do a reduction starting from the "Hamilton Cycle" problem for directed graphs which is known to be NP-complete [9]. Given an instance of a "Hamilton Cycle", that is, a directed graph $G = (V, E)$, let N be the cardinality of V and assume that $V = \{1, \ldots, N\}$. We construct an instance of a partially asynchronous process as follows: we let $\mathcal{X} = \{u_0, u_1, \ldots, u_N\}$ and $x^* = u_0$. We also let $K = M = N$ and construct the operators T_k by letting $T_j(u_i) = u_j$, if $(i, j) \in E$, and $T_j(u_i) = u_0$, otherwise.

Suppose that there exists a Hamiltonian cycle in the graph G. Equivalently, there exists a permutation $\pi_1, \ldots, \pi_N$ of the nodes of G such that $(\pi_i, \pi_{i+1}) \in E$ (for $i = 1, \ldots, N-1$) and $(\pi_N, \pi_1) \in E$. Let $s(i) = \pi_i$, for $i = 1, \ldots, n$. The periodic extension of s into an infinite sequence clearly belongs to $\mathcal{S}_M$. Moreover, the trajectory corresponding to this execution sequence cycles through the elements $u_1, \ldots, u_n$ of $\mathcal{X}$ and, therefore, never reaches x^* and the process does not converge.

Conversely, suppose that there exists some sequence $s \in \mathcal{S}_M$ for which the co -esponding trajectory x^s does not converge to x^*. Since $K = M$, it follows that for every M steps each operator has to be applied exactly once. Because of the way that the operators were defined, different operators lead to different states. Consequently, every state other than x^* is visited exactly once every M steps. This implies that there exists a Hamilton cycle in G. This is a polynomial time reduction (of the complement) of the Hamilton cycle problem to our problem and proves the first assertion of the theorem.

Suppose now that $M \geq 2NK$. We claim that the partially asynchronous process converges if and only if the totally asynchronous process converges. (The result then follows because the convergence of the totally asynchronous process may be tested in polynomial time, see Section 4.) The "if" direction is trivial. For the reverse direction, suppose that the totally asynchronous process does not converge. Consider the colored graph of Section 4. By Theorem 4.1, there exists some state $x_0 \neq x^*$ and a cycle which goes through x_0 and uses all colors. It follows that there exists some state x_0 such that, for every color k, there exists a cycle through x_0 which uses color k. For every color k this cycle may be chosen

to be of length at most $2N$. (We need at most $N-1$ steps to reach any state which is reachable from x_0). We now merge together the cycles corresponding to the different colors to produce a cycle of length at most $2NK$ which uses all colors. Hence the partially asynchronous process does not converge if $M = 2NK$ and clearly cannot converge if M has any larger value.

We now consider the problem for instances in which K is fixed to some value. Because of the result proved in the last paragraph we can and will assume that $M < 2NK$. Given a process $(\mathscr{X}, x^*, T_1, \ldots, T_K)$, we consider a new process defined on the space $\mathscr{Y} = \mathscr{X} \times \{1, \ldots, M\}^K$ and involving K operators $Q_1, \ldots, Q_K$. We choose the operators $Q_1, \ldots, Q_K$ so that $Q_i(x, m_1, \ldots, m_K) = (x^*, 1, \ldots, 1)$ if $x = x^*$ or if $m_k = M$ for some $k \neq i$. Also, $Q_i(x, m_1, \ldots, m_K) = (T_i x, m_1', \ldots, m_K')$, where $m_k' = 1$ if $i = k$ and $m_k' = m_k + 1$ if $k \neq i$ and $m_k < M$. That is, for any execution sequence $s \in \mathscr{S}_M$ the first component of y is the same as the state x of the original iterative process; the other components of y record how far in the past each operator was applied for the last time. The latter components serve to detect whether an execution sequence does not belong to $\mathscr{S}_M$; if it does not, the y process is forced to converge. It should now be clear that there exists some sequence $s \in \mathscr{S}_M$ and some $x_0 \in \mathscr{X}$ such that $x^s(n, x_0)$ does not converge to x^* if and only if there exists some sequence $s \in \mathscr{S}$ such that $y^s(n, (x_0, 1, \ldots, 1))$ does not converge to $(x^*, 1, \ldots, 1)$. However, the latter statement is equivalent to the absence of cycles (not visiting x^*) in the graph representing the transitions of the y process. This may be tested in time polynomial in the size of that graph; that is, in time polynomial in NM^K. Now recall that K is fixed and that $M < 2NK$, to conclude that we have an algorithm which is polynomial in N. □

We study next partially asynchronous processes when $\mathscr{X}$ is a finite-dimensional Euclidean space and each T_k is a linear operator (i.e., a matrix). (Notice that the distributed gradient algorithm of Section 3 is of this form when f is a quadratic cost function.)

Theorem 6.2. *Given a set $T_1, \ldots, T_K$ of square matrices and an integer M, the problem of deciding whether the corresponding partially asynchronous process converges is NP-hard.*

Proof. Given an asynchronous process $P = (\mathscr{X}, x^*, T_1, \ldots, T_K)$ on the finite state space $\mathscr{X} = \{x^*, u_1, \ldots, u_n\}$, we consider the n-dimensional Euclidean space R^n. We identify x^* with the origin and each u_i with the ith unit vector e_i. We define the matrices $T_1', \ldots, T_K'$ by requiring that $T_k'(e_i) = 0$, if $T_k(x_i) = x^*$, and $T_k'(e_i) = e_j$, if $T_k(u_i) = u_j$. This establishes a complete correspondence between the two processes. Clearly, the second process, when initialized at any unit vector, converges if and only if the original process on the finite state space converges. Moreover, the second process converges, starting from an an arbitrary initial element of R^n, if and only if it converges when it is initialized at a unit vector, because of linearity. In view of Theorem 6.1, the proof is complete. □

Incidentally, the proofs of Theorems 6.1 and 6.2 lead to the following corollary.

Corollary 6.1. *Given a set* $\{T_1, \ldots, T_K\}$ *of square matrices, the problem of deciding whether the product* $T_{\pi_1} T_{\pi_2} \cdots T_{\pi_K}$ *is an asymptotically stable matrix, for every permutation* π *of the set* $\{1, \ldots, K\}$, *is NP-hard.*

The above results are useful not so much because one might want to devise an algorithm to test for the convergence of partially asynchronous processes, but, rather, because they imply that there are no simple (i.e., efficiently testable) necessary and sufficient conditions for convergence. Furthermore, unless $NP =$ co-NP (which is considered unlikely) the problem of recognizing convergent partially asynchronous processes (on a finite state space) does not belong to NP.[6] This implies that some partially asynchronous processes are convergent but there is no concise (i.e., polynomial) certificate which testifies to this; in other words, exponentially long proofs may be required. All these are in sharp contrast to totally asynchronous processes and show that an analog of the theory of Section 5 does not exist. (Clearly, processes with infinite state spaces cannot be any easier to analyze than processes with finite state spaces, unless of course a special structure is introduced.) As far as applications are concerned, it seems that the best that can be done is to develop useful sufficient conditions for convergence and these should not be expected to be tight, in general.

7. Conclusions

We have studied the structure and the associated conditions for convergence of two classes of asynchronous processes. The general conclusion that may be drawn is that a Lyapunov-type theory applies to totally asynchronous iterative processes: such processes are convergent if and only if a Lyapunov function (appropriately defined) testifies to this. Of course, as is the case in ordinary Lyapunov stability theory, the existence result is not very helpful when one is actually confronted with the problem of constructing such a Lyapunov function. Nevertheless, distributed algorithms are typically designed with some kind of Lyapunov function in mind. In fact, the results of Sections 4 and 5 suggest that a meaningful procedure for designing distributed algorithms is to specify a suitable Lyapunov function first and then try to construct operators which decrease its value. After all, this methodology is fairly common in certain areas of systems theory.

Even though the existence of a Lyapunov function which demonstrates convergence is a nonconstructive result, we have seen that for finite state spaces such functions may be constructed in polynomial time. However, for partially asynchronous algorithms this is not the case (unless $P = NP$). This suggests that partially asynchronous algorithms on infinite state spaces are also qualitatively

[6] This is because in our proofs we reduced the complement of an NP-complete problem, that is, a co-NP-complete problem, to the problem of recognizing convergent partially asynchronous processes.

different and harder to analyze from their totally asynchronous counterparts. A methodology based on Lyapunov functions is not universal in the context of partially asynchronous algorithms and each particular algorithm may require special techniques in order to have its convergence established.

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