NONEXPANSIVE PIECEWISE CONSTANT HYBRID SYSTEMS ARE CONSERVATIVE *

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Abstract. Consider a partition of $\mathbb{R}^n$ into finitely many polyhedral regions $\mathcal{D}_i$ and associated drift vectors $\mu_i \in \mathbb{R}^n$. We study "hybrid" dynamical systems whose trajectories have a constant drift, $\dot{x} = \mu_i$, whenever x is in the interior of the ith region $\mathcal{D}_i$, and behave consistently on the boundary between different regions. Our main result asserts that if such a system is *nonexpansive* (i.e., if the Euclidean distance between any pair of trajectories is a nonincreasing function of time), then the system must be *conservative*, i.e., its trajectories are the same as the trajectories of the negative subgradient flow associated with a potential function. Furthermore, this potential function is necessarily convex, and is linear on each of the regions $\mathcal{D}_i$. We actually establish a more general version of this result, by making seemingly weaker assumptions on the dynamical system of interest.

Key words. Dynamical systems, nonexpansive systems, conservative systems, subgradient flow, piecewise constant system, piecewise linear potential

1. Introduction. In this paper we study "hybrid" dynamical systems with the following special structure. Consider a partition of $\mathbb{R}^n$ into finitely many polyhedral regions $\mathcal{D}_i$, and associated drift vectors $\mu_i \in \mathbb{R}^n$. We focus on systems whose trajectories obey $\dot{x} = \mu_i$ whenever x is in the interior of the ith region $\mathcal{D}_i$, and refer to them as *polyhedral hybrid systems*.

Polyhedral systems arise in many different contexts in which the dynamics are relatively simple, with a finite set of possible control actions,and different actions applied in different regions of the state space. Examples include communication networks [10, 4], processing systems [6], manufacturing systems, and inventory management [5, 3]. More concretely, this type of systems describes the fluid model dynamics of several policies for real-time job scheduling [4] that choose at each time a service vector out of a finite set of possible such vectors, based on the current system state (i.e., the queue lengths), as in the celebrated Max-Weight algorithm [10] and its generalizations [2]. See Fig. 1 for a simple example.

Our main result asserts that if a polyhedral hybrid system is *nonexpansive* (i.e., if the Euclidean distance between any pair of trajectories is a nonincreasing function of time) then the system must be *conservative*, i.e., its trajectories are the same as the trajectories of the negative subgradient flow associated with a potential function. Furthermore, this potential function can be chosen to be piecewise linear, with finitely many linear pieces, and convex; finally, the potential function is linear on each of the regions $\mathcal{D}_i$.

Once we establish that the system is conservative, the particular properties of the potential function are not too surprising. For example, it is not hard to show that if a dynamical system is conservative and nonexpansive, then the underlying potential function must be convex.[1] The converse also turns out to be true: the (negative)

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[1] We provide the proof for the special case of a conservative system $\dot{x} = -\nabla\Phi(x)$ associated with a *differentiable* potential function Φ. Consider a pair x and y of points in $\mathbb{R}^n$, and let $x(\cdot)$ and $y(\cdot)$

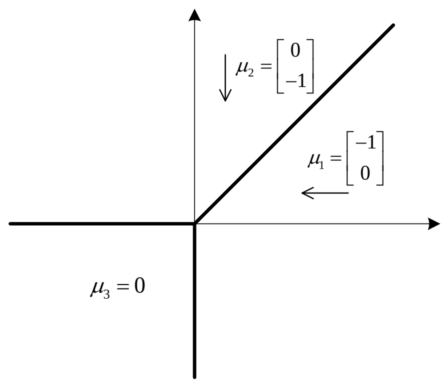

FIG. 1. *The dynamics of a network with two queues and a server that serves a longest queue at each time, with arbitrary time-sharing in case of a tie. Moreover, if both queues are negative, the server goes idle. There are three polyhedral regions, and there is a constant drift vector, μ_i, in the interior of each region. Furthermore at the boundary between two (or three) regions, the direction of motion is a convex combination of the drifts μ_i in the adjacent regions. Such a system is nonexpansive and can also be described as the negative subgradient flow associated with the piecewise linear convex function* $\Phi(x) = \max\big(-\mu_1^T x,\, -\mu_2^T x,\, 0\big)$.

subgradient flow associated with a convex potential function is nonexpansive.[2]

The unexpected part of our result is that a nonexpansive polyhedral system is conservative in the first place. This is quite surprising because a nonexpansive system is not necessarily conservative,[3] and these two properties appear to be quite unrelated. In general, a system being conservative is intimately tied with a curl-free property of the underlying field. On the other hand, nonexpansive systems have no reason to be curl-free. It is thus remarkable that for the systems that we consider the combination of nonexpansiveness and the polyhedral structure imposes a combinatorial structure around points where regions intersect, which then translates to a curl-free condition.

We note that we actually establish our main result for a seemingly broader class of systems: well-formed nonexpansive finite-partition systems; cf. Definitions 2.2 and 2.4, without assuming, for example, that the different regions are polyhedral. It so happens that the nonexpansiveness assumption together with well-formedness forces the regions to be convex; and a partition of $\mathbb{R}^n$ into convex regions forces the regions to be polyhedral. Thus, this broader class of systems in fact reduces to the class of polyhedral hybrid systems introduced in the beginning of this section.

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be two trajectories initialized at x and y, respectively. Then, the nonexpansive property implies that

$$2 \cdot (x-y)^T \big(\nabla\Phi(x) - \nabla\Phi(y)\big) \;=\; -\frac{d}{dt}\big\|x(t)-y(t)\big\|^2 \;\geq\; 0. \tag{1.1}$$

Thus, Φ has an increasing directional derivative over the line segment that starts at y and ends at x. Since x and y are arbitrary, it follows that Φ is convex.

[2]For the special case where the convex potential function $\Phi(\cdot)$ is differentiable, the argument is as follows. For any pair $x(\cdot)$ and $y(\cdot)$ of trajectories and any time t, it follows from the convexity of $\Phi(\cdot)$ that

$$\frac{d}{dt}\big\|x(t)-y(t)\big\|^2 \;=\; -2\cdot\big(x(t)-y(t)\big)^T\Big(\nabla\Phi\big(x(t)\big) - \nabla\Phi\big(y(t)\big)\Big) \leq\; 0, \tag{1.2}$$

and the system is nonexpansive.

[3]A simple example is a system in $\mathbb{R}^2$ whose trajectories are circles, traversed with a fixed angular velocity.

The rest of the paper is organized as follows. In Section 2, we lay out the setting of interest and present the main result, whose proof is then given in Section 3. In the course of the proof, we use several lemmas, whose proofs are relegated to the appendices for improved readability. Finally, we discuss the implications of our result, along with some open problems, in Section 4.

2. Definitions and Main Result. In this section, we define some terminology and three classes of systems. We then present our main result, which states that under a nonexpansiveness assumption, these three classes are equivalent.

2.1. Definitions. We start with a definition of general dynamical systems, following [8].

Definition 2.1 (Dynamical Systems). *A* dynamical system *is a set-valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. A* trajectory *of a system F is a function $x(\cdot)$ whose derivative, denoted by $\dot{x}$, exists and satisfies the differential inclusion $\dot{x}(t) \in F\big(x(t)\big)$, except possibly for a measure-zero set of times.*[4]

For general systems, some assumptions are needed in order to guarantee existence of trajectories, which leads us to the next definition.

Definition 2.2 (Well-Formed Dynamical System). *A dynamical system F is well-formed, if:*

a) $F(\cdot)$ is upper semicontinuous.[5]

b) There exists a constant α such that $\big\|F(x)\big\| \leq \alpha\big(1 + \|x\|\big)$, for all $x \in \mathbb{R}^n$.

c) For every $x \in \mathbb{R}^n$, $F(x)$ is a closed and convex set.

The second condition above prevents solutions from blowing up in finite time. The other two conditions are technical, but are often satisfied.

In this paper, we focus on nonexpansive systems, defined below.

Definition 2.3 (Nonexpansive Systems). *A dynamical system F is called* nonexpansive *if for any pair $x(\cdot)$ and $y(\cdot)$ of trajectories and for any times t_1 and t_2, with $t_2 \geq t_1$,*

$$\big\|x(t_2) - y(t_2)\big\| \leq \big\|x(t_1) - y(t_1)\big\|, \tag{2.1}$$

where $\|\cdot\|$ stands for the Euclidean norm.

Note that the nonexpansiveness property automatically guarantees that if a solution exists, then it is unique. We next define the most general class of systems to be considered.

Definition 2.4 (Finite-Partition Systems). *A dynamical system F is said to be a* finite-partition *system if there exists a finite collection of distinct vectors $\mu_1, \ldots, \mu_m$, such that for any $x \in \mathbb{R}^n$, there is some i for which $\mu_i \in F(x)$.*

Loosely speaking, Definition 2.4 requires that there be a finite set of "special vectors" μ so that at each point in the state space, at least one of these vectors is a possible drift, in the sense of $\dot{x} = \mu$. As we will see later, a finite-partition system induces a natural "tiling" of $\mathbb{R}^n$, a concept that we will define shortly. We will then

[4] The solution concept used here to described trajectories is consistent with the definition of (unperturbed) trajectories in [7].

[5] A set-valued function F is called upper semicontinuous if for every $x \in \mathbb{R}^n$ and any open subset $\mathcal{V}$ of $\mathbb{R}^n$ such that $F(x) \subseteq \mathcal{V}$, there exists an open neighbourhood $\mathcal{U}$ of x such that $F(y) \subseteq \mathcal{V}$, for all $y \in \mathcal{U}$.

proceed with a formal definition of the class of polyhedral hybrid systems, which was already discussed in Section 1. Throughout the paper, we use the term "polyhedron" to refer to a closed and convex set that can be defined in terms of finitely many linear inequalities.

DEFINITION 2.5 (Polyhedral tiling). *A collection $\mathcal{D}_i$, $i = 1, \ldots, m$, of nonempty polyhedral subsets of $\mathbb{R}^n$ ("regions") is said to be a* polyhedral tiling *if it satisfies the following:*

a) $\bigcup_i \mathcal{D}_i = \mathbb{R}^n$*;*
b) each region $\mathcal{D}_i$ has a nonempty interior;
c) if $i \neq j$, then $\mathcal{D}_i$ and $\mathcal{D}_j$ have disjoint interiors.

Given a polyhedral tiling, one can consider hybrid systems that have a constant drift in the interior of each polyhedron. In what follows, we use the notation Conv(A) to denote the convex hull of a set $A \subseteq \mathbb{R}^n$.

DEFINITION 2.6 (Polyhedral Hybrid Systems). *Consider a polyhedral tiling $\mathcal{D}_1, \ldots, \mathcal{D}_m$, and associated* distinct *vectors $\mu_1, \ldots, \mu_m \in \mathbb{R}^n$. The corresponding* polyhedral hybrid system *F is defined by letting $F(x) = \mathrm{Conv}\big(\{\mu_i \mid x \in \mathcal{D}_i\}\big)$, for every $x \in \mathbb{R}^n$. In particular, if x belongs to the interior of $\mathcal{D}_i$, then $F(x) = \{\mu_i\}$.*

As will be discussed in Section 4, the requirement that the vectors μ_i are distinct can be made without loss of generality.

The third class of systems that we will study is comprised of *conservative* systems, that move along the negative subgradient of a potential field. When we further restrict the structure of the potential, we obtain the class of systems that was studied in [7].

DEFINITION 2.7 (FPCS Systems).

a) A function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called finitely piecewise linear and convex *if it is of the form $\Phi(x) = \max_i \big(-\mu_i^T x + b_i\big)$, for some finite set of distinct pairs $(\mu_i, b_i) \in \mathbb{R}^n \times \mathbb{R}$, $i = 1, \ldots, m$.*
b) We say that F is a Finitely Piecewise Constant Subgradient (FPCS) *system if there exists a finitely piecewise linear convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that for any $x \in \mathbb{R}^n$, $F(x) = -\partial\Phi(x)$, where $\partial\Phi(x)$ denotes the subdifferential of Φ at x.*

FPCS systems are of particular interest, as they arise in various contexts. On the technical side they have some unusually strong sensitivity properties: as shown in [7], the effect of an additive external perturbation on the state is bounded above by a constant times the magnitude of the *integral* of the perturbation (as opposed to the integral of the magnitude, which is a much weaker upper bound).

2.2. Main Result. We are interested in the relation between the following three classes of systems: (i) well-formed finite-partition systems, (ii) polyhedral hybrid systems, and (iii) FPCS systems. FPCS systems are automatically polyhedral hybrid systems, but the converse is not always true. Furthermore, finite-partition systems are much more general; in particular, they include systems where the regions of constant drift are not polyhedral. Nevertheless, our main result states that once we restrict our attention to nonexpansive systems, these three classes are essentially the same.

THEOREM 2.8. *Let F be a nonexpansive dynamical system.*

a) F is an FPCS system if and only if it is a polyhedral hybrid system.
b) If F is a polyhedral hybrid system, or, equivalently, an FPCS system, then it is a well-formed finite-partition system.

c) If F is a well-formed finite partition system, then there exists a polyhedral hybrid system (or, equivalently, an FPCS system) F' such that for any initial state $x(0)$, the two systems follow the same trajectory. Furthermore, $F'(x) \subseteq F(x)$, for all $x \in \mathbb{R}^n$.

While some parts of the result are straightforward, Part (c) is rather surprising. Nonexpansive systems are in general not conservative. Nevertheless, nonexpansive polyhedral hybrid systems turn out to be be conservative because they are implicitly forced to have a special and subtle structure at the intersection of different regions. In the same spirit, nonexpansiveness, the finite-partition property, and well-formedness, taken together, force a polyhedral structure on the regions involved.

On the technical side, Theorem 2.8 establishes equivalence of finite-partition and FPCS systems, in terms of the trajectories that they generate, even though the mappings F may be a bit different. In order to appreciate what is involved here, consider a one-dimensional system of the following form:

$$F(x) = \begin{cases} 1, & \text{if } x < 0, \\ [-2, 2], & \text{if } x = 0, \\ -1, & \text{if } x > 0. \end{cases}$$

Trajectories of this system move at unit speed towards the origin; once at the origin, trajectories remain there. We observe that F is upper semicontinuous and that this is a well-formed nonexpansive finite-partition system, with two drift vectors, $\mu_1 = -1$ and $\mu_2 = 1$. Yet, it is not a polyhedral hybrid system because $F(0)$ is strictly larger than the convex hull of the set $\{-1, 1\}$ of drifts in the regions adjacent to 0. In the same spirit, the trajectories of F are the negative subgradient flow for the convex potential function $\Phi(x) = |x|$, but, strictly speaking, F is not a FPCS system, because $F(0) = [-2, 2] \neq [-1, 1] = -\partial\Phi(0)$. On the other hand, the system F is "equivalent" to the FPCS system associated with $\Phi(x) = |x|$ (and also a polyhedral hybrid system), in the sense that they generate the same trajectories.

3. Proof of Theorem 2.8.

In this section, we give the proof of Theorem 2.8. Throughout, and for any statement that we make about various systems, we always assume that the systems are nonexpansive, even if this assumption is not explicitly stated.

3.1. Part (a): Polyhedral hybrid systems and FPCS systems are equivalent.

We begin with the difficult direction of Part (a): we show that nonexpansive polyhedral hybrid dynamical systems are conservative. The proof is given in a sequence of several lemmas, whose proofs are relegated to appendices. The general line of argument involves an explicit construction of a convex potential function associated with a given polyhedral hybrid system. To develop our construction, we first consider geometric polygonal paths that travel from one region to another. We endow these paths with certain weights and show that cycles have zero weight. We then leverage this conservation property to associate a weight with each region. We finally use these region weights to define a convex potential function over $\mathbb{R}^n$, and show that its subdifferential flow yields the same trajectories as the given polyhedral hybrid system.

We now start with the formal proof. Consider a polyhedral tiling $\mathcal{D}_1, \ldots, \mathcal{D}_m$, distinct vectors $\mu_1, \ldots, \mu_m$, and the associated polyhedral hybrid system

$$F(x) = \mathrm{Conv}\big(\{\mu_i \mid x \in \mathcal{D}_i\}\big). \tag{3.1}$$

We say that two regions $\mathcal{D}_i$ and $\mathcal{D}_j$, are *adjacent* if $i \neq j$ and their intersection is nonempty. To any pair of adjacent regions $\mathcal{D}_i$ and $\mathcal{D}_j$ we associate a weight b_{ij}, by defining

$$b_{ij} \triangleq \sup_{x \in \mathcal{D}_i} \left(\mu_i - \mu_j\right)^T x. \tag{3.2}$$

We also let $b_{ii} = 0$, for all i.

LEMMA 3.1 (Local Conservation of Weights). *The supremum in (3.2) is always attained, and every b_{ij} is finite. Furthermore, the weights b_{ij} have the following properties:*

a) $b_{ij} = -b_{ji}$, whenever $\mathcal{D}_i$, $\mathcal{D}_j$ are adjacent;
b) $b_{ij} + b_{jk} + b_{ki} = 0$, for all triples i, j, k for which $\mathcal{D}_i \bigcap \mathcal{D}_j \bigcap \mathcal{D}_k \neq \emptyset$.

The proof is given in Appendix A and relies on the nonexpansive property of the dynamics.

In our next step, we associate to each region $\mathcal{D}_i$ a weight b_i, such that $b_i - b_j = b_{ij}$, for all pairs of adjacent regions $\mathcal{D}_i$ and $\mathcal{D}_j$. This is made possible because of a global conservation property of weights over geometric paths on the tiling. Before going through this global conservation property, we need an auxiliary lemma and some definitions. Lemma 3.2 below essentially asserts that if we can find points that come sufficiently close, simultaneously, to each one of three polyhedra, then these polyhedra must have a common element. Note that such a property is special to polyhedra, and does not hold more generally: it is not hard to find examples of disjoint closed and convex sets whose distance is zero.

LEMMA 3.2 (A Geometric Property of Polyhedral Tilings). *Consider the regions $\mathcal{D}_i$ in a polyhedral tiling. There exists a constant $\gamma > 0$ such that if a closed Euclidean ball of radius γ intersects with any (not necessarily distinct) three regions $\mathcal{D}_i$, $\mathcal{D}_j$, and $\mathcal{D}_k$, then $\mathcal{D}_i \bigcap \mathcal{D}_j \bigcap \mathcal{D}_k$ is nonempty.*

The proof is given in Appendix B, and relies on a bound derived through an auxiliary linear program. We now fix the constant γ of Lemma 3.2, and let

$$\delta = \gamma/3. \tag{3.3}$$

DEFINITION 3.3 (Paths and Cycles). *A sequence $x_1, \ldots, x_t$ of points of $\mathbb{R}^n$ is a* **path** *if each x_i is in the interior of some region $\mathcal{D}_{k_i}$. A path is* **jump-free** *if for $i = 1, \ldots, t-1$, the regions $\mathcal{D}_{k_i}$ and $\mathcal{D}_{k_{i+1}}$ are either the same or adjacent. A path is* **fine** *if for $i = 1, \ldots, t-1$, and with δ as defined in (3.3), we have $\left\|x_i - x_{i+1}\right\| \leq \delta$. To every jump-free path, we associate a weight:*

$$W\left(x_1, \ldots, x_t\right) = \sum_{i=1}^{t-1} b_{k_i k_{i+1}}. \tag{3.4}$$

A **cycle** *is a path that "ends where it started", i.e., $x_1 = x_t$.*

Note that, as a consequence of Lemma 3.2 and our choice of δ, every fine path is jump-free.[6] The next lemma establishes a global conservation property for path weights.

[6]To see this, consider x_i and x_{i+1} on a fine path. The segment that joins x_i to x_{i+1} is contained in a ball of radius at most δ, and which intersects both $\mathcal{D}_{k_i}$ and $\mathcal{D}_{k_{i+1}}$. Then, Lemma 3.2 applies and shows that $\mathcal{D}_{k_i}$ and $\mathcal{D}_{k_{i+1}}$ must intersect, and are therefore the same or adjacent.

Lemma 3.4 (Global Conservation of Weights). *Every fine cycle has zero weight.*

The proof is given in Appendix C. In the proof, we first leverage Lemmas 3.1 and 3.2 to show that every fine cycle of "small size" (i.e., contained inside a small ball) has zero weight. We then represent a given fine cycle as a superposition of multiple fine cycles of smaller sizes, and use induction.

Using Lemma 3.4, we can now associate a weight b_i to each region $\mathcal{D}_i$, as follows. Let $b_1 = 0$. For any other region $\mathcal{D}_i$, let b_i be the weight of an arbitrary fine path $x_1, \ldots, x_t$, with $x_1 \in \mathcal{D}_i$ and $x_t \in \mathcal{D}_1$. Then, using Lemma 3.4, it is not hard to see that b_i is independent of the choice of the fine path, and is therefore well-defined. Moreover, for any pair of adjacent regions $\mathcal{D}_i$ and $\mathcal{D}_j$, we have

$$b_{ij} = b_i - b_j. \tag{3.5}$$

We now define a potential function $\Phi : \mathbb{R}^n \to \mathbb{R}$, by letting

$$\Phi(x) = \max_i \, (-\mu_i^T x + b_i). \tag{3.6}$$

Lemma 3.5 (Properties of Φ). *For any $x \in \mathbb{R}^n$, we have $\Phi(x) = -\mu_i^T x + b_i$ if and only if $x \in \mathcal{D}_i$.*

The proof is given in Appendix D and relies on a delicate interplay between the definition of weights b_i and b_{ij} and the properties of polyhedral tilings. Furthermore, it also relies on the requirement that the vectors μ_i be distinct; cf. Definition 2.6; the lemma fails to hold otherwise.

Let us fix some $x \in \mathbb{R}^n$. Since Φ is convex, it has a subdifferential at x, and

$$-\,\partial\Phi(x) \;=\; \mathrm{Conv}\left\{\mu_i \,\middle|\, i \in \mathrm{argmax}_j \,(-\mu_j^T x + b_j)\right\} \;=\; \mathrm{Conv}\left\{\mu_i \mid x \in \mathcal{D}_i\right\} \;=\; F(x). \tag{3.7}$$

where the first equality is from the subdifferential formula for the pointwise maximum functions [1, Section 3.1.1], the second equality follows from Lemma 3.5, and the last equality is the definition of F. Therefore, the polyhedral hybrid system F coincides with the FPCS system associated with the particular potential function Φ that we introduced.

This completes the proof of one direction in Part (a) of the Theorem. The proof of the converse statement, that FPCS systems are polyhedral hybrid systems, is fairly straightforward. It only requires us to check a few details, which is done in Appendix E.

3.2. Part (b): Polyhedral hybrid (or FPCS) systems are well-formed finite-partition systems. This result is just a simple observation. The details are as follows. Consider a polyhedral hybrid system F, with $F(x) = \mathrm{Conv}\big(\{\mu_i \mid x \in \mathcal{D}_i\}\big)$, and fix some particular x. Recall that every polyhedral region $\mathcal{D}_j$ is closed and that $\bigcup_j \mathcal{D}_j = \mathbb{R}^n$. Therefore, $\mathcal{U} \triangleq \mathbb{R}^n \setminus \bigcup_{j \,:\, x \notin \mathcal{D}_j} \mathcal{D}_j$ is an open neighbourhood of x contained in $\bigcup_{i \,:\, x \in \mathcal{D}_i} \mathcal{D}_i$. It follows that for every $y \in \mathcal{U}$, we have $\{\mu_i \mid y \in \mathcal{D}_i\} \subseteq \{\mu_i \mid x \in \mathcal{D}_i\}$, and as a result, $F(y) = \mathrm{Conv}\big(\{\mu_i \mid y \in \mathcal{D}_i\}\big) \subseteq \mathrm{Conv}\big(\{\mu_i \mid x \in \mathcal{D}_i\}\big) = F(x)$. Therefore, F is upper semicontinuous. The bound on $\|F(x)\|$ is immediate because $F(x)$ is a subset of the convex hull of $\mu_1, \ldots, \mu_m$. Finally, each $F(x)$ is the convex hull of finitely many vectors, and is therefore closed and convex. It follows that the system is well-formed. Furthermore, each $F(x)$ contains at least one of the vectors μ_i, which implies that we have a well-formed finite-partition system.

Given that FPCS systems are equivalent to hybrid systems, FPCS systems are also well-formed finite-partition systems, and this concludes the proof of Part (b).

3.3. Part (c): A finite-partition system has the same trajectories as a polyhedral hybrid system. In this subsection, we show that any well-formed nonexpansive finite-partition system F is associated with a polyhedral hybrid system F' that has the same trajectories. The key step involves showing that under the nonexpansiveness assumption, the regions associated with a finite-partition system are essentially polyhedral.

We fix a well-formed and nonexpansive finite-partition dynamical system F and an associated finite collection of distinct vectors $\mu_1, \ldots, \mu_m$ such that for any $x \in \mathbb{R}^n$, there exists some μ_i for which $\mu_i \in F(x)$. We define a collection of regions,

$$\mathcal{R}_i \triangleq \left\{x \in \mathbb{R}^n \,\middle|\, \mu_i \in F(x)\right\}, \qquad i = 1, \ldots, m. \tag{3.8}$$

Our first lemma asserts that these regions are closed.

LEMMA 3.6. *The set $\mathcal{R}_i$ is closed, for every i.*

The proof is based on the well-formedness of F, and is given in Appendix F. We now consider another collection $\mathcal{D}_1 \ldots, \mathcal{D}_m$ of regions, defined by

$$\mathcal{D}_i \triangleq \text{closure}\big(\mathcal{R}_i^\circ\big), \quad i = 1, \ldots, m, \tag{3.9}$$

where $\mathcal{R}_i^\circ$ is the interior of $\mathcal{R}_i$. Then, Lemma 3.6 implies that

$$\mathcal{D}_i \subseteq \mathcal{R}_i, \quad i = 1, \ldots, m. \tag{3.10}$$

The regions $\mathcal{R}_i$ can be fairly unstructured, For example, they may have multiple isolated points. On the other hand, the regions $\mathcal{D}_i$ are much better behaved, as stated in the next lemma. We let $\mathcal{I}$ be the set of indices for which $\mathcal{D}_i$ is nonempty (equivalently, $\mathcal{R}_i$ has nonempty interior).

LEMMA 3.7 (Regions $\mathcal{D}_i$ Form a Polyhedral Tiling). *The regions $\mathcal{D}_i$, $i \in \mathcal{I}$, defined in* (3.9)*, form a polyhedral tiling.*

The proof is given in Appendix G.

We now define a system F', by letting, $F'(x) = \text{Conv}\left\{\mu_i \,\middle|\, x \in \mathcal{D}_i,\, i \in \mathcal{I}\right\}$, for every x. From Lemma 3.7, the regions $\mathcal{D}_i$ form a polyhedral tiling. As a result, F' is a polyhedral hybrid system. On the other hand, (3.10) implies that if $x \in \mathcal{D}_i$, then $x \in \mathcal{R}_i$, i.e., $\mu_i \in F(x)$. Since F is well-formed, $F(x)$ is convex. Therefore, $F(x)$ contains the convex hull $\text{Conv}\left\{\mu_i \,\middle|\, x \in \mathcal{D}_i\right\}$, and we have established that $F'(x) \subseteq F(x)$, for all $x \in \mathbb{R}^n$, as desired.

To complete the proof of Part (d), it remains to show that the two systems, F and F', have the same sets of trajectories. Since $F'(x) \subseteq F(x)$, for all x, it follows that any trajectory of F' is also a trajectory of F. For the reverse direction, let $x(\cdot)$ be a trajectory of F, initialized at some $x(0)$. Let $y(\cdot)$ be a trajectory of F' with the same initial condition, $y(0) = x(0)$. (General existence results for well-formed systems guarantee that such a trajectory $y(\cdot)$ exists; cf. Theorem 4.3 of [8]). The inclusion $F' \subseteq F$ implies that $y(\cdot)$ is also a trajectory of F. However, since F is nonexpansive, it has a unique trajectory with initial condition $x(0)$. Therefore, $y(t) = x(t)$, for all $t \geq 0$, and $x(\cdot)$ is also a trajectory of F'. This completes the proof of Part (c) of the theorem.

4. Discussion. In this section, we discuss our result and its implications. We also discuss some open problems and directions for future research.

4.1. Our Result. We have established that nonexpansive polyhedral hybrid systems are conservative, and in particular they follow the (negative) subgradient flow of a piecewise linear convex potential function, with finitely many pieces. We also showed that the same is true for a seemingly more general class of finite-partition systems. One consequence of our results is that previously established upper bounds on the sensitivity of FPCS systems to additive external perturbations [7] now carry over to nonexpansive polyhedral hybrid systems.

The finiteness of the number of constant-drift regions is central to this result, because in its absence, a well-formed nonexpansive dynamical system need not be conservative (cf. Footnote 3). In this respect, our result is quite counter-intuitive: every smooth dynamical system over a compact domain can be approximated by a polyhedral hybrid system, with arbitrarily high accuracy over a bounded time interval. Hence, our result might suggest that every nonexpansive smooth system must be conservative, but this is not the case. This apparent contradiction is resolved by observing that such a finite approximation will not in general preserve the nonexpansiveness property.

Our result is analogous to a corollary of Stokes' theorem [9], that any curl-free vector field is conservative. In our context, the curl-free property is replaced by a local conservation property of nonexpansive polyhedral hybrid systems; cf. Lemma 3.1(b). What is somewhat surprising however is that there is nothing in our assumptions that suggests such a curl-free property. Instead, this property emerges through the delicate interplay between the finiteness and the nonexpansiveness assumptions. In the same spirit, the polyhedral shape of the different regions in a finite-partition system is not assumed but emerges in an unexpected way from seemingly unrelated assumptions.

4.2. Open Problems. Our result suggests several possible generalizations and open problems for future research, which we list below.

a) Does the result generalize to the case where the set $\mathcal{S}$ in the definition of finite-partition systems (Definition 2.4) is allowed to be countably infinite? Or if we assume that we have a polyhedral tiling with a countably infinite number of regions $\mathcal{D}_i$, would some additional assumptions be needed?

b) What if we relax the constant drift property in the interior of each region? More concretely, consider a nonexpansive dynamical system F, and a polyhedral tiling of $\mathbb{R}^n$ and assume that the restriction of F to each region is a (negative) subgradient field. Does this imply that F is a (negative) subgradient field?

c) What if we consider a finite-partition system defined only on a convex (but not necessarily polyhedral) subset of $\mathbb{R}^n$? We conjecture that the result remains valid, but a different proof seems to be needed.

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Appendix A. Proof of Lemma 3.1. We consider a polyhedral hybrid system F, as in (3.1), and the weights defined in (3.2). We start with the proof of Part (a) of the lemma.

Let $\mathcal{D}_i^\circ$ stand for the interior of $\mathcal{D}_i$, which is assumed nonempty for all i; cf. Definition 2.5(b). Let $x_i \in \mathcal{D}_i^\circ$ and $x_j \in \mathcal{D}_j^\circ$, and let $x_i(\cdot)$ and $x_j(\cdot)$ be two trajectories with initial conditions $x_i(0) = x_i$ and $x_j(0) = x_j$. Then, at time $t = 0$, we have $\dot{x}_i(0) = \mu_i$ and $\dot{x}_j(0) = \mu_j$. Using the nonexpansive property of the dynamics, we obtain

$$\begin{aligned} \big(\mu_i - \mu_j\big)^T \big(x_i - x_j\big) \;&=\; \big(\dot{x}_i(0) - \dot{x}_j(0)\big)^T \big(x_i(0) - x_j(0)\big) \\ &=\; \frac{1}{2} \cdot \frac{d^+}{dt} \big\|x_i(t) - x_j(t)\big\|^2 \Big|_{t=0} \\ &\leq\; 0. \end{aligned} \tag{A.1}$$

By taking the limit in (A.1) along sequences in $\mathcal{D}_i^\circ$ and $\mathcal{D}_j^\circ$ that converge to x_i and x_j, respectively, we conclude that (A.1) holds for every $x_i \in \mathcal{D}_i$ and $x_j \in \mathcal{D}_j$.

Consider two adjacent regions $\mathcal{D}_i$ and $\mathcal{D}_j$, and a point x in their intersection. Since $x \in \mathcal{D}_j$, it follows from (A.1) that if $x_i \in \mathcal{D}_i$, then $\big(\mu_i - \mu_j\big)^T x_i \leq \big(\mu_i - \mu_j\big)^T x$. Therefore,

$$b_{ij} \;=\; \sup_{x_i \in \mathcal{D}_i} \big(\mu_i - \mu_j\big)^T x_i \;\leq\; \big(\mu_i - \mu_j\big)^T x < \infty. \tag{A.2}$$

On the other hand, since $x \in \mathcal{D}_i$,

$$b_{ij} \;=\; \sup_{x_i \in \mathcal{D}_i} \big(\mu_i - \mu_j\big)^T x_i \;\geq\; \big(\mu_i - \mu_j\big)^T x. \tag{A.3}$$

Therefore, for any $x \in \mathcal{D}_i \bigcap \mathcal{D}_j$,

$$b_{ij} \;=\; \big(\mu_i - \mu_j\big)^T x. \tag{A.4}$$

Switching the roles of i and j, we obtain

$$b_{ji} \;=\; \big(\mu_j - \mu_i\big)^T x \;=\; -\big(\mu_i - \mu_j\big)^T x \;=\; -b_{ij}. \tag{A.5}$$

This establishes the first part of the lemma.

For the second part, consider a triple i, j, k, such that $\mathcal{D}_i \bigcap \mathcal{D}_j \bigcap \mathcal{D}_k \neq \emptyset$. If two of these indices are equal, e.g., if $i = j \neq k$, then the first part of the lemma implies that

$$b_{ij} + b_{jk} + b_{ki} = b_{ii} + b_{ik} + b_{ki} = b_{ik} + b_{ki} = 0.$$

If the three indices are distinct, then fix some $x \in \mathcal{D}_i \bigcap \mathcal{D}_j \bigcap \mathcal{D}_k$. It follows from (A.4) that

$$b_{ij} + b_{jk} + b_{ki} \;=\; \left(\mu_i - \mu_j\right)^T x + \left(\mu_j - \mu_k\right)^T x + \left(\mu_k - \mu_i\right)^T x \;=\; 0. \tag{A.6}$$

This completes the proof of the lemma.

Appendix B. Proof of Lemma 3.2. Consider three regions $\mathcal{D}^1$, $\mathcal{D}^2$, and $\mathcal{D}^3$, with empty intersection, i.e., $\mathcal{D}^1 \bigcap \mathcal{D}^2 \bigcap \mathcal{D}^3 = \emptyset$. Since the regions $\mathcal{D}_i$, $i = 1, 2, 3$, are polyhedra, there exist matrices A^i, all rows of which have unit norm, and vectors b^i such that

$$\mathcal{D}^i = \left\{ x \in \mathbb{R}^n \,\middle|\, A^i x - b^i \preceq 0 \right\}, \qquad i = 1, 2, 3, \tag{B.1}$$

where $\preceq$ stands for componentwise inequality.

Consider the linear programming problem with variables ϵ and x, of minimizing ϵ subject to

$$A^i x - b^i \preceq \epsilon \mathbb{1}^i, \quad i = 1, 2, 3, \tag{B.2}$$

where $\mathbb{1}^i$ is a vector with the same dimension as b^i and with all entries equal to one. For any $x \in \mathbb{R}^n$, there is a large enough ϵ such that (ϵ, x) is a feasible point of (B.2). Therefore, (B.2) has a nonempty set of feasible points. Moreover, since $\mathcal{D}^1 \bigcap \mathcal{D}^2 \bigcap \mathcal{D}^3$ is empty, for any feasible point (ϵ, x) of (B.2), we have $\epsilon > 0$. Given that we are dealing with a linear programming problem with a nonempty feasible region and with finite optimal cost, the optimal value of the objective is attained, and must be a positive number, which we denote by ϵ^*. Hence, for any $x \in \mathbb{R}^n$, the constraint $A^i x - b^i \preceq (\epsilon^*/2)\mathbb{1}^i$ is violated, for at least one $i \in \{1, 2, 3\}$. Since the rows of A have unit norm, it follows that the closed $(\epsilon^*/2)$-neighbourhood of x does not intersect $\mathcal{D}^i$. The desired result follows by letting γ be equal to the minimum of the constants $\epsilon^*/2$ over all triples $\mathcal{D}^1, \mathcal{D}^2, \mathcal{D}^3$ that have empty intersection.

Appendix C. Proof of Lemma 3.4. We will use induction on s to establish the following sequence of statements, for $s = 3, 4, \ldots$:

H_s : "Every fine cycle contained in a Euclidean ball of radius $s\delta$, has zero weight".

Base case ($s = 3$): Consider a fine cycle $x_1, \ldots, x_t$, and suppose that it is contained in the closed 3δ-neighbourhood of a point $y \in \mathbb{R}^n$. Let $\mathcal{D}_{k_0}$ be a region that contains y. Since $3\delta = \gamma$, Lemma 3.2 implies that $\mathcal{D}_{k_0} \bigcap \mathcal{D}_{k_i} \bigcap \mathcal{D}_{k_{i+1}}$ is nonempty, for $i = 1, \ldots, t-1$. Then, it follows from Lemma 3.1 that

$$b_{k_0 k_i} + b_{k_i k_0} = 0, \quad i = 1, \ldots, n-1, \tag{C.1}$$

$$b_{k_0 k_i} + b_{k_i k_{i+1}} + b_{k_{i+1} k_0} = 0, \quad i = 1, \ldots, n-1. \tag{C.2}$$

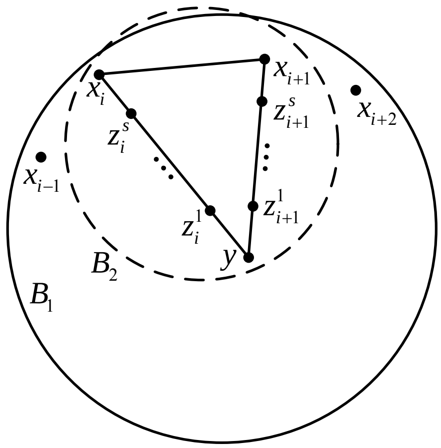

FIG. 2. *Decomposition of a fine cycle $x_1, \ldots, x_t$ into smaller fine cycles, in the proof of Lemma 3.4. Here the cycle $x_1, \ldots, x_t$ is contained in a ball B_1 of radius $s\delta$. The figure shows one of the cycles in the decomposition $y, z_i^1, \ldots, z_i^s, x_i, x_{i+1}, z_{i+1}^s, \ldots, z_{i+1}^1, y$, which is contained in a smaller ball B_2, of radius at most $(s-1)\delta$.*

Therefore,

$$\begin{aligned} W(x_1, \ldots, x_t) &= \sum_{i=1}^{t-1} b_{k_i k_{i+1}} \\ &= \sum_{i=1}^{t-1} \Big(b_{k_0 k_i} + b_{k_i k_{i+1}} + b_{k_i k_0} \Big) \\ &= \sum_{i=1}^{t-1} \Big(b_{k_0 k_i} + b_{k_i k_{i+1}} + b_{k_{i+1} k_0} \Big) \\ &= 0, \end{aligned}$$

where the equalities follow from the definition of the weight $W(\cdot)$, (C.1), $x_n = x_1$, and (C.2), respectively.

Induction step: We now fix some $s \geq 4$ and assume the induction hypothesis that H_{s-1} is true. We will show that H_s is also true, and do this by decomposing a fine cycle into fine cycles of smaller radii. Consider a fine cycle $x_1 \ldots, x_t$ that is contained in a ball of radius $s\delta$. Let y be a point such that $\|y - x_i\| < (s+1)\delta$, for $i = 1, \ldots, t-1$. For any $i \leq t-1$, consider a sequence $z_i^1, \ldots, z_i^s$ of equidistant points on the line segment between y and x_i; see Fig. 2 for an illustration. In view of the freedom in the choice of y, we can and will assume that y is in the interior of some region $\mathcal{D}_{k_0}$ and each z_j^i is in the interior of a region $\mathcal{D}_{k_j^i}$. Then, for any $i \leq t-1$, the sequence

$$\xi^i = \big(y, z_i^1, \ldots, z_i^s, x_i, x_{i+1}, z_{i+1}^s, \ldots, z_{i+1}^1, y\big)$$

is a fine cycle.

Moreover, the triangle with vertices y, x_i, and x_{i+1} has two edges of length at most $(s+1)\delta$ and one edge, $x_i x_{i+1}$, of length at most δ. Therefore, this triangle is contained in a closed ball of radius

$$\frac{(s+1)\delta + \delta}{2} = \left(\frac{s}{2} + 1\right) \delta \leq (s-1)\delta, \tag{C.3}$$

where the inequality holds because $s \geq 4$. Then, the induction hypothesis H_{s-1} implies that the cycle ξ^i has zero weight.

Consider now a cycle ξ which is the concatenation of the cycles $\xi^1, \ldots, \xi^{t-1}$. The weight of ξ equals the sum of the weights of the cycles ξ^i, and is therefore zero. Finally, note that ξ consists of the arcs of the original cycle $x_1, \ldots, x_{t-1}$, together with the intermediate paths that join y to x_i, for the different i. For every i, the path from y to x_i is traversed twice, in opposite directions. Hence, using the property $b_{ij} + b_{ji} = 0$ of the weights, the total contribution of these intermediate paths is zero. It follows that the weight of the original cycle $x_1, \ldots, x_{t-1}$ is equal to the weight of ξ, which is zero.

Appendix D. Proof of Lemma 3.5.

We consider the function Φ defined by $\Phi(x) = \max_i (-\mu_i^T x + b_i)$. We first prove the "if" direction of the result. That is, we fix some $x \in \mathcal{D}_i$ and show that $\Phi(x) = -\mu_i^T x + b_i$; equivalently, that

$$-\mu_i^T x + b_i \geq -\mu_j^T x + b_j, \qquad \text{for all } x \in \mathcal{D}_i \text{ and all } j. \tag{D.1}$$

Fix some $j \neq i$, and a point $y \in \mathcal{D}_j$. Given that $\mathcal{D}_j$ has nonempty interior, which is disjoint from $\mathcal{D}_i$, we can and will assume that $y \neq x$. By tracing the different regions crossed by the segment xy, it is not hard to see that we can find a finite sequence of *distinct* points $x_1, \ldots, x_k$ and associated regions $\mathcal{D}^1, \ldots, \mathcal{D}^k$ such that:

(i) $x_1 = x$ and $x_k = y$;
(ii) $x_l \in \mathcal{D}^l$, for $l = 1, \ldots, k$; in particular, $\mathcal{D}^1 = \mathcal{D}_i$ and $\mathcal{D}^k = \mathcal{D}_j$;
(ii) the regions $\mathcal{D}^l$ and $\mathcal{D}^{l+1}$ are adjacent, for $l = 1, \ldots, k-1$;

We let μ^l and b^l be the drift vector and weight, respectively, associated with region $\mathcal{D}^l$. Then, for $l = 1, \ldots, k-1$, we have $x_l \in D^l$, and it follows from (3.5) and (3.2) that

$$b^l - b^{l+1} = \sup_{z \in \mathcal{D}^l} \left(\mu^l - \mu^{l+1}\right)^T z \geq \left(\mu^l - \mu^{l+1}\right)^T x_l.$$

Equivalently,

$$-\left(\mu^l - \mu^{l+1}\right)^T x_l + \left(b^l - b^{l+1}\right) \geq 0. \tag{D.2}$$

Interchanging the roles of i_l and i_{l+1}, we obtain

$$-\left(\mu^{l+1} - \mu^l\right)^T x_{l+1} + \left(b^{l+1} - b^l\right) \geq 0.$$

Adding the last two inequalities, we get

$$-\left(\mu^l - \mu^{l+1}\right)^T \left(x_l - x_{l+1}\right) \geq 0.$$

Since x_{l+1} lies further along the line segment that connects x_1 to x_k, we see that $x_l - x_{l+1}$ is a *positive* multiple of $x_1 - x_l$. Consequently, if $l < k$, then

$$-\left(\mu^l - \mu^{l+1}\right)^T \left(x_1 - x_l\right) \geq 0. \tag{D.3}$$

Adding (D.2) and (D.3), we obtain

$$-\left(\mu^l - \mu^{l+1}\right)^T x_1 + \left(b^l - b^{l+1}\right) \geq 0.$$

It follows that

$$\begin{aligned}
-\mu_i^T x + b_i &= -(\mu^1)^T x_1 + b^1 \\
&= \sum_{l=1}^{k-1} \Big(- \big(\mu^l - \mu^{l+1}\big)^T x_1 + \big(b^l - b^{l+1}\big)\Big) - (\mu^k)^T x_1 + b^k \\
&\geq -(\mu^k)^T x_1 + b^k \\
&= -\mu_j^T x + b_j.
\end{aligned}$$

Recall that x is an arbitrary element of $\mathcal{D}_i$ and that j is an arbitrary index. This establishes Eq. (D.1) and concludes the proof of the "if" direction of the lemma.

We now establish the converse ("only if") direction. We have already shown that for $x \in \mathcal{D}_i$, i is a maximizing index in the definition of Φ. The proof of the converse starts by establishing a stronger statement: in the interior of $\mathcal{D}_i$, the maximizing index is unique. (The assumption that the μ_i are distinct will have to be invoked here.)

Claim D.1. *Fix some $x \in \mathbb{R}^n$. The index i attains the maximum (over j) in the expression* $\max_j (-\mu_j^T x + b_j)$ *and is the unique maximizer if and only if $x \in \mathcal{D}_i^\circ$.*

Proof of the Claim. To establish the "only if" direction, suppose that i is the unique maximizer of $\max_j (-\mu_j^T x + b_j)$; equivalently $-\mu_j^T x + b_j < \Phi(x)$, for all $j \neq i$. Then, Eq. (D.1) implies that $x \notin \mathcal{D}_j$, for all $j \neq i$. Since $\bigcup_j \mathcal{D}_j = \mathbb{R}^n$ and each $\mathcal{D}_j$ is closed, it follows that $x \in \mathbb{R}^n \setminus \bigcup_{j \neq i} \mathcal{D}_j = \mathcal{D}_i^\circ$.

To establish the "if" direction, fix an index i and a point $x \in \mathcal{D}_i^\circ$. Take an arbitrary $j \neq i$ and a small enough ϵ such that $y \triangleq x + \epsilon(\mu_i - \mu_j) \in \mathcal{D}_i$. Then,

$$\begin{aligned}
-\mu_i^T x + b_i &= -\mu_i^T y + b_i + \epsilon \mu_i^T (\mu_i - \mu_j) \\
&\geq -\mu_j^T y + b_j + \epsilon \mu_i^T (\mu_i - \mu_j) \\
&= -\mu_j^T x + b_j + \epsilon (\mu_i - \mu_j)^T (\mu_i - \mu_j) \\
&= -\mu_j^T x + b_j + \epsilon \big\|\mu_i - \mu_j\big\|^2 \\
&> -\mu_j^T x + b_j,
\end{aligned} \tag{D.4}$$

where the first inequality is due to (D.1) after interchanging the indices i and j, and the last inequality holds because the μ_i are distinct. Since Eq. (D.4) holds for every $j \neq i$, it follows that i is the unique maximizer of $\max_j (-\mu_j^T x + b_j)$, and the claim follows. ∎

Suppose now that $\Phi(x) = -\mu_i^T x + b_i$; in particular,

$$- \mu_i^T x + b_i \geq -\mu_j^T x + b_j, \tag{D.5}$$

for all j. We need to show that $x \in \mathcal{D}_i$. From the definition of a polyhedral tiling, $\mathcal{D}_i$ has nonempty interior. Fix an arbitrary $y \in \mathcal{D}_i^\circ$. Claim D.1 implies that for any $j \neq i$,

$$- \mu_i^T y + b_i > -\mu_j^T y + b_j. \tag{D.6}$$

For $\epsilon \in (0,1)$, let $z_\epsilon \triangleq (1-\epsilon)x + \epsilon y$. It follows from Eqs. (D.5) and (D.6) that for any $\epsilon \in (0,1)$,

$$- \mu_i^T z_\epsilon + b_i > -\mu_j^T z_\epsilon + b_j, \qquad \forall\, j \neq i. \tag{D.7}$$

Employing Claim D.1 once again, it follows from (D.7) that for any $\epsilon \in (0,1)$, $z_\epsilon \in \mathcal{D}_i^\circ$. Thus, as ϵ goes to zero, z_ϵ converges to x from within the interior of $\mathcal{D}_i$. Since $\mathcal{D}_i$ is closed, we conclude that $x \in \mathcal{D}_i$, and the lemma follows.

Appendix E. Proof of the converse direction of Part (a) of Theorem 2.8. Consider an FPCS system, with $F(x) = -\partial\Phi(x)$, where

$$\Phi(x) = \max_{i=1,\dots,m} (-\mu_i^T x + b_i),$$

and where the pairs (μ_i, b_i) are distinct.

For any i, let $\mathcal{D}_i = \{x \mid \Phi(x) = -\mu_i^T x + b_i\}$, which is a polyhedron. Clearly, the union of the sets $\mathcal{D}_i$ is all of $\mathbb{R}^n$. Let $\mathcal{I}$ be the set of all i for which the polyhedron $\mathcal{D}_i$ has nonempty interior. Then, $\mathbb{R}^n \setminus \bigcup_{i\in\mathcal{I}} \mathcal{D}_i$ is an open subset of the empty interior set $\bigcup_{i\notin\mathcal{I}} \mathcal{D}_i$, and is therefore empty. It follows that $\bigcup_{i\in\mathcal{I}} \mathcal{D}_i = \mathbb{R}^n$. Also note that for any two regions $\mathcal{D}_i$ and $\mathcal{D}_j$, their interiors $\mathcal{D}_i^\circ = \{x \mid -\mu_i^T x + b_i > -\mu_k^T x + b_k,\, \forall k \neq i\}$ and $\mathcal{D}_j^\circ = \{x \mid -\mu_j^T x + b_j > -\mu_k^T x + b_k,\, \forall k \neq j\}$ are disjoint. This proves that the sets $\mathcal{D}_i$, $i \in \mathcal{I}$, comprise a polyhedral tiling.

Let $\Phi'(x) = \max_{i\in I} (-\mu_i^T x + b_i)$. Then, $\Phi'(x) \leq \Phi(x)$, for all x. On the other hand, since $\bigcup_{i\in\mathcal{I}} \mathcal{D}_i = \mathbb{R}^n$, any $x \in \mathbb{R}^n$ lies in $\mathcal{D}_i$ for some $i \in \mathcal{I}$. Thus, for $x \in \mathcal{D}_i$, we have $\Phi(x) = -\mu_i^T x + b_i \leq \max_{j\in\mathcal{I}} (-\mu_j^T x + b_j) = \Phi'(x)$. We conclude that, $\Phi'(x) = \Phi(x)$, for all $x \in \mathbb{R}^n$.

Note that if $\mu_j = \mu_i$ and (without loss of generality) $b_i < b_j$, then $-\mu_i^T x + b_i < -\mu_j^T x + b_j$ for all x, so that i cannot attain the maximum in the definition of Φ, and $i \notin \mathcal{D}_i$. This shows that when we restrict to indices i in $\mathcal{I}$, the corresponding vectors μ_i must be distinct, as required in the definition of polyhedral hybrid systems.

Finally, it follows form the subdifferential formula for the pointwise maximum of linear functions [1, Section 3.1.1] that $-\partial\Phi(x) = -\partial\Phi'(x) = \mathrm{Conv}\big(\{\mu_i \mid x \in \mathcal{D}_i,\, i \in \mathcal{I}\}\big)$. Therefore, $-\partial\Phi(x)$ is the polyhedral hybrid system associated with the regions $\mathcal{D}_i$ and the vectors μ_i, for $i \in \mathcal{I}$.

Appendix F. Proof of Lemma 3.6. Let us consider a region $\mathcal{R}_i$. We will prove the equivalent statement that the complement of $\mathcal{R}_i$ is open. Let us fix some $x \notin \mathcal{R}_i$. From the definition of $\mathcal{R}_i$, the assumption $x \notin \mathcal{R}_i$ implies that $\mu_i \notin F(x)$. Moreover, since F is well-formed, $F(x)$ is closed. Therefore, there exists an open subset $\mathcal{V}$ of $\mathbb{R}^n$ such that $F(x) \subseteq \mathcal{V}$ and $\mu_i \notin \mathcal{V}$. It then follows from the upper-semicontinuity of $F(\cdot)$ that there exists an open neighbourhood $\mathcal{U}$ of x such that $F(y) \subseteq \mathcal{V}$, for all $y \in \mathcal{U}$. Therefore, $\mu_i \notin F(y)$, for all $y \in \mathcal{U}$. Equivalently, $y \notin \mathcal{R}_i$, for all $y \in \mathcal{U}$. Hence, for any x in the complement of $\mathcal{R}_i$, there exists an open neighborhood of x which is contained in the complement of $\mathcal{R}_i$. This shows that the complement of $\mathcal{R}_i$ is open, and concludes the proof.

Appendix G. Proof of Lemma 3.7. We will show that the regions $\mathcal{D}_i \triangleq \mathrm{closure}\big(\mathcal{R}_i^\circ\big)$, $i = 1, \dots, m$, defined in (3.9), and restricting to those regions that are nonempty (i.e., with $i \in \mathcal{I}$) form a tiling.

From the finite-partition property, for every x there exists some i such that $\mu_i \in F(x)$. It follows that every x must belong to some $\mathcal{R}_i$ (cf. Equation (3.8)), and

$\bigcup_i \mathcal{R}_i = \mathbb{R}^n$. Thus,

$$\begin{aligned} \mathbb{R}^n \setminus \Big(\bigcup_i \mathcal{D}_i \Big) &= \Big(\bigcup_i \mathcal{R}_i \Big) \setminus \Big(\bigcup_i \mathcal{D}_i \Big) \\ &\subseteq \bigcup_i \left(\mathcal{R}_i \setminus \mathcal{D}_i \right) \\ &\subseteq \bigcup_i \left(\mathcal{R}_i \setminus \mathcal{R}_i^\circ \right). \end{aligned} \tag{G.1}$$

Each $\mathcal{R}_i \setminus \mathcal{R}_i^\circ$ has empty interior. It follows that $\bigcup_i \left(\mathcal{R}_i \setminus \mathcal{R}_i^\circ \right)$ has empty interior as well. From the definition $\mathcal{D}_i = \text{closure}\big(\mathcal{R}_i^\circ\big)$, each set $\mathcal{D}_i$ is automatically closed. Hence $\mathbb{R}^n \setminus \big(\bigcup_i \mathcal{D}_i \big)$ is open. At the same time, from (G.1), the latter set is contained in the empty-interior set $\bigcup_i \left(\mathcal{R}_i \setminus \mathcal{R}_i^\circ \right)$. This implies that $\mathbb{R}^n \setminus \big(\bigcup_i \mathcal{D}_i \big)$ is empty, and therefore

$$\bigcup_i \mathcal{D}_i = \mathbb{R}^n. \tag{G.2}$$

Note that for every i, $\mathcal{D}_i \supseteq \mathcal{R}_i^\circ$. Thus, any interior point of $\mathcal{R}_i$ is also an interior point of $\mathcal{D}_i$. In particular, if $\mathcal{D}_i$ has empty interior, then $\mathcal{R}_i^\circ$ is empty, and so is its closure, $\mathcal{D}_i$. It follows that whenever $\mathcal{D}_i$ is nonempty (formally, when $i \in \mathcal{I}$), $\mathcal{D}_i$ has nonempty interior. From now on, we restrict attention to nonempty regions, $\mathcal{D}_i$, i.e., with $i \in \mathcal{I}$.

If there is a single nonempty region $\mathcal{D}_i$, then that region is all of $\mathbb{R}^n$ and is therefore closed and convex. Suppose now that there are multiple nonempty regions, and let us fix some distinct $i, j \in \mathcal{I}$. Let $x_i \in \mathcal{D}_i^\circ$ and $x_j \in \mathcal{D}_j^\circ$, and let $x_i(\cdot)$ and $x_j(\cdot)$ be two trajectories with initial conditions $x_i(0) = x_i$ and $x_j(0) = x_j$. It can be seen that for small positive times t, $x_i + \mu_i t$ is a possible trajectory; in fact, because of the nonexpansive property, it is the unique trajectory. It follows that at time $t = 0$, we have $\dot{x}_i(0) = \mu_i$ and, similarly, $\dot{x}_j(0) = \mu_j$. Therefore,

$$\begin{aligned} \big(\mu_i - \mu_j\big)^T \big(x_i - x_j\big) &= \big(\dot{x}_i(0) - \dot{x}_j(0)\big)^T \big(x_i(0) - x_j(0)\big) \\ &= \frac{1}{2} \cdot \frac{d^+}{dt} \big\| x_i(t) - x_j(t) \big\|^2 \Big|_{t=0}. \end{aligned}$$

Using the nonexpansive property of the dynamics, we conclude that

$$\big(\mu_i - \mu_j\big)^T \big(x_i - x_j\big) \;\le\; 0. \tag{G.3}$$

Since $\mathcal{D}_i = \text{closure}\big(\mathcal{R}_i^\circ\big) = \text{closure}\big(\mathcal{D}_i^\circ\big)$, for any $x_i \in \mathcal{D}_i$, there exists a sequence of points in the interior of $\mathcal{D}_i$ that converges to x_i. For any $x_i \in \mathcal{D}_i$ and $x_j \in \mathcal{D}_j$, the inequality (G.3) holds along such sequences of points converging to x_i and x_j. Hence, (G.3) also holds for every $x_i \in \mathcal{D}_i$ and $x_j \in \mathcal{D}_j$.

Consider again some distinct $i, j \in \mathcal{I}$, and let

$$d_{ij} \;\triangleq\; \inf_{x_j \in \mathcal{D}_j} \big(\mu_i - \mu_j\big)^T x_j. \tag{G.4}$$

Let us fix some $x_i \in \mathcal{D}_i$. It follows from (G.3) that as x_j ranges over $\mathcal{D}_j$, the expression $\big(\mu_i - \mu_j\big)^T x_j$ is lower bounded by $\big(\mu_i - \mu_j\big)^T x_i$. It follows that the infimum in Eq. (G.4), and therefore d_{ij} as well is finite. In particular, this justifies the second equality in the following calculation:

$$d_{ij} + d_{ji} = \inf_{x_j \in \mathcal{D}_j} \big(\mu_i - \mu_j\big)^T x_j + \inf_{x_i \in \mathcal{D}_i} \big(\mu_j - \mu_i\big)^T x_i \tag{G.5}$$

$$= \inf_{\substack{x_i \in \mathcal{D}_i \\ x_j \in \mathcal{D}_j}} \big(\mu_i - \mu_j\big)^T \big(x_j - x_i\big) \tag{G.6}$$

$$= - \sup_{\substack{x_i \in \mathcal{D}_i \\ x_j \in \mathcal{D}_j}} \big(\mu_i - \mu_j\big)^T \big(x_i - x_j\big) \tag{G.7}$$

$$\geq 0, \tag{G.8}$$

where the inequality is due to (G.3).

For every $i \in \mathcal{I}$, we define the polyhedron

$$\mathcal{P}_i = \bigcap_{j \neq i,\, j \in \mathcal{I}} \big\{ x \mid \big(\mu_i - \mu_j\big)^T x \leq d_{ij} \big\}. \tag{G.9}$$

Suppose that $x_i \in \mathcal{D}_i$. Then, for every $j \in \mathcal{I}$, with $j \neq i$, and any $x_j \in \mathcal{D}_j$, (G.3) yields $(\mu_i - \mu_j)^T x_i \leq (\mu_i - \mu_j)^T x_j$. Taking the infimum over all $x_j \in \mathcal{D}_j$, and using the definition of d_{ij}, we obtain $(\mu_i - \mu_j)^T x_i \leq d_{ij}$. Since this is true for every such j, and comparing with the definition of $\mathcal{P}_i$, we obtain

$$\mathcal{D}_i \subseteq \mathcal{P}_i, \tag{G.10}$$

for all i. It then follows from (G.2) that $\bigcup_i \mathcal{P}_i = \mathbb{R}^n$.

Suppose again that $i, j \in \mathcal{I}$, with $j \neq i$. If x belongs to the interior of $\mathcal{P}_i$, none of the inequality constraints in the definition of $\mathcal{P}_i$ can be satisfied with equality, and therefore,

$$\big(\mu_i - \mu_j\big)^T x \,<\, d_{ij}. \tag{G.11}$$

On the other hand, from (G.4), if $x \in \mathcal{P}_j$, then $\big(\mu_i - \mu_j\big)^T x \geq d_{ij}$, which contradicts (G.11). Therefore,

$$\mathcal{P}_i^\circ \bigcap \mathcal{P}_j = \emptyset. \tag{G.12}$$

We conclude that the polyhedra $\mathcal{P}_i$ have disjoint interiors and comprise a polyhedral tiling.

We now show that $\mathcal{P}_i \subseteq \mathcal{D}_i$. We have

$$\mathcal{P}_i^\circ \,\subseteq\, \mathbb{R}^n \backslash \bigcup_{j \neq i,\, j \in I} \mathcal{P}_j \,\subseteq\, \mathbb{R}^n \backslash \bigcup_{j \neq i,\, j \in I} \mathcal{D}_j \,\subseteq\, \mathcal{D}_i, \tag{G.13}$$

where the inclusions follow from (G.12), (G.10), and (G.2), respectively. Therefore,

$$\mathcal{P}_i \,=\, \text{closure}(\mathcal{P}_i^\circ) \,\subseteq\, \text{closure}(\mathcal{D}_i) \,=\, \mathcal{D}_i. \tag{G.14}$$

It then follows from (G.14) and (G.10) that $\mathcal{D}_i = \mathcal{P}_i$, for all i, and the regions $\mathcal{D}_i$ thereby comprise a polyhedral tiling.