

# Pole-Zero Phase Maps

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## INTRODUCTION

Determining the phase of the loop transfer function  $L(s)$  at arbitrary points on the  $s$ -plane is an important skill for students in introductory control subjects. Evans showed that the magnitude and phase of a transfer function can be determined from the pole-zero map using simple vector geometry [1], [2]. For example, the magnitude and phase of a transfer function of the form

$$L(s) = \frac{s - z}{s - p}$$

at a particular value of  $s = s_1$  can be evaluated from the vectors shown in Figure 1. The complex number  $s_1 - z$  in the numerator is represented by the vector from the point  $s = z$  to the point  $s = s_1$ . The length  $r_z$  of this vector is the magnitude of  $s_1 - z$ , and the angle  $\phi_z$  that this vector makes with respect to the real axis is the phase of  $s_1 - z$ . Writing both vectors of complex numbers in polar form yields

$$L(s_1) = \frac{s_1 - z}{s_1 - p} = \frac{r_z e^{j\phi_z}}{r_p e^{j\phi_p}} = \frac{r_z}{r_p} e^{j(\phi_z - \phi_p)}$$

or, simply,

$$|L(s_1)| = \frac{r_z}{r_p}, \quad \angle L(s_1) = \phi_z - \phi_p.$$

Despite the simplicity of this calculation, students often have some initial difficulty visualizing the effects of poles and zeros on the phase of  $L(s)$  throughout the  $s$ -plane. Using Matlab to produce arrow plots, the phase of  $L(s)$  can be displayed on a pole-zero map. For example, Figure 2 shows the phase of the single-pole transfer function

$$L(s) = \frac{1}{s + 1}.$$

The blue arrows point in the direction of the phase of  $L(s)$ . Students can verify the vector-geometry arguments above by exploring a few test points.

## ACCURATE PHASE CHARACTERISTICS ON BODE PLOTS

Once students grow accustomed to these phase maps,  $s$ -plane relationships leading to the development of Bode plots and Evans root-locus plots can be demonstrated. For example, Figure 3 shows that the two poles at  $s = -1$  of

$$L(s) = \frac{1}{s(s + 1)^2}$$

cause a slowly varying phase characteristic on the imaginary axis, while Figure 4 shows that the two lightly damped complex-conjugate poles of

$$L(s) = \frac{1}{s(s^2 + 0.2s + 1)}$$

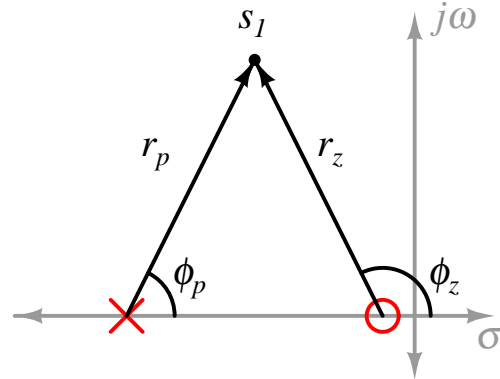


Fig. 1. Vector geometry for finding the magnitude and phase of  $L(s_1)$ . The quantity  $s_1 - p$  is represented by a vector whose length is  $r_p$  and which has the angle  $\phi_p$ . The magnitude of  $L(s_1)$  is equal to  $r_z/r_p$ , and the phase of  $L(s_1)$  is equal to  $\phi_z - \phi_p$ .

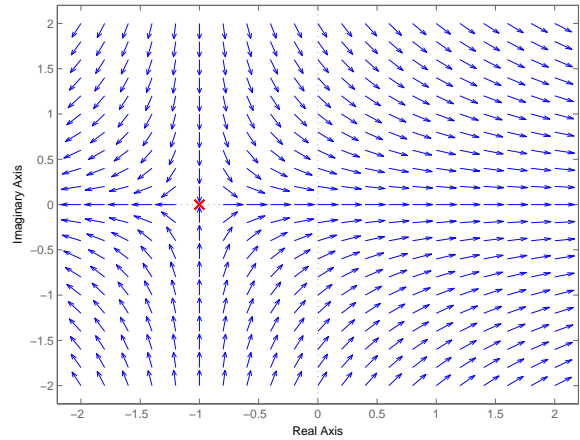


Fig. 2. Pole-zero phase map of  $L(s) = 1/(s + 1)$ . The transfer-function pole is shown at  $s = -1$ . The blue arrows point in the direction of the phase of  $L(s)$  for the value of  $s$  at the base of the arrow. For example, the phase of  $L(s)$  is zero degrees for values of  $s$  on the positive real axis.

create a much sharper transition in the phase of the frequency response. This distinction is evident in the Bode plots shown in Figure 5. The pole-zero phase map helps students visualize the cause of the sharper phase transition of the lightly damped poles.

## FINDING THE EVANS ROOT LOCUS

The branches of the Evans root locus are found where the phase of the loop transfer function is an odd integer multiple of  $-180^\circ$ . Satisfying this root-locus angle condition for

$$L(s) = \frac{K}{s(s^2 + 0.2s + 1)}$$

is accomplished by finding the blue arrows that point to the left in Figure 4. Figure 6 shows the Evans root-locus plot for

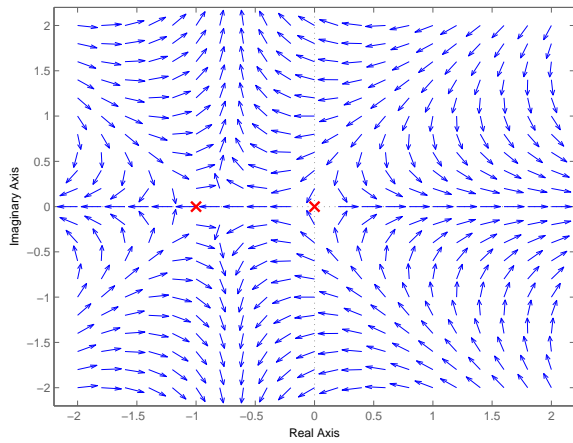


Fig. 3. Pole-zero phase map of  $L(s) = 1/s(s+1)^2$ . With all poles on the real axis, the phase of  $L(j\omega)$  along the imaginary axis changes slowly from  $-90^\circ$  at the origin to  $-270^\circ$  for large positive values of  $\omega$ .

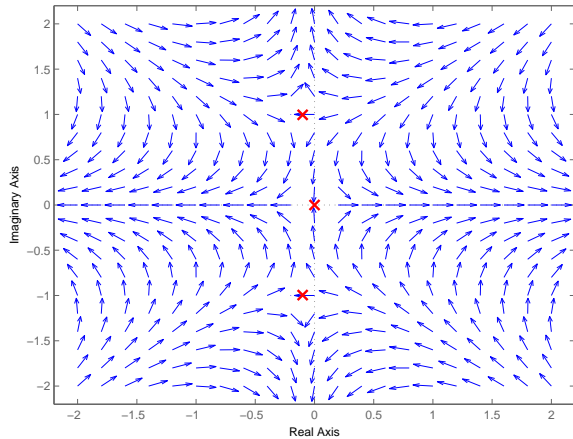


Fig. 4. Pole-zero phase map of  $L(s) = 1/s(s^2 + 0.2s + 1)$ . The phase of  $L(j\omega)$  along the imaginary axis changes quickly from  $-90^\circ$  to  $-270^\circ$  near the locations of the lightly damped complex-conjugate poles.

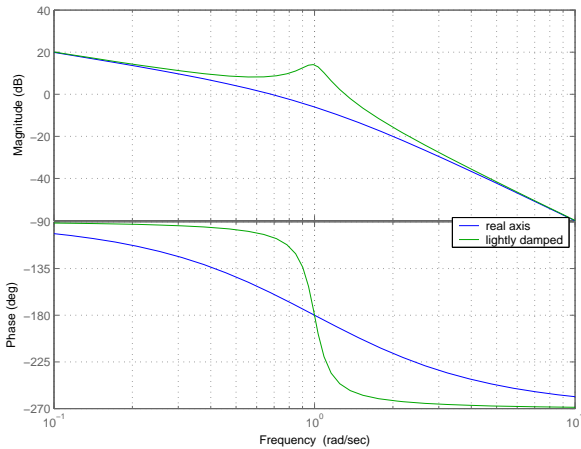


Fig. 5. Bode plots of  $L(s) = 1/s(s+1)^2$  (blue) and  $L(s) = 1/s(s^2 + 0.2s + 1)$  (green). The phase transition for the latter transfer function is much sharper due to the complex-conjugate poles close to the imaginary axis, as shown in Figure 4.

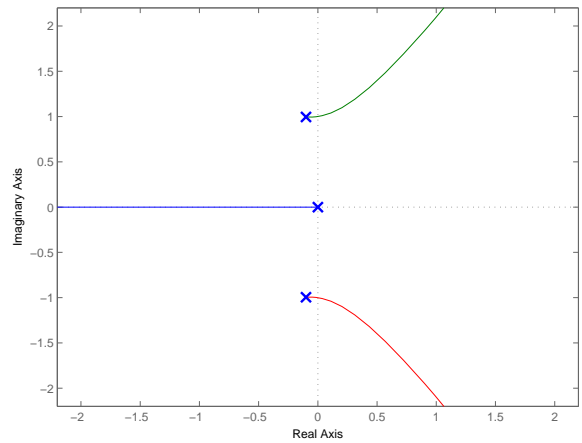


Fig. 6. Evans root-locus plot of  $L(s) = K/s(s^2 + 0.2s + 1)$ . The branches of the root locus lie where the phase of  $L(s)$  is  $-180^\circ$ , which corresponds to the arrows pointing left in Figure 4.

this transfer function, tracing the paths of the blue arrows that point to the left.

### DEMONSTRATING COMPENSATION

Pole-zero phase maps can help students appreciate the interrelation of classical-control analysis tools and measures of relative stability. For example, the effect of lead compensation on phase margin and the centroid of the asymptotes of the root locus can be seen on a single pole-zero phase map. Consider the double integrator

$$L(s) = \frac{K}{s^2}.$$

The phase of the frequency response is always  $-180^\circ$ , as shown in Figure 7. This system can be stabilized with a lead compensator yielding the loop transfer function

$$L(s) = \frac{K}{s^2} \left( \frac{s+1}{0.25s+1} \right),$$

which provides a positive phase margin. Examination of the pole-zero phase map in Figure 8 reveals that the compensated system has phase that approaches  $-140^\circ$  along the positive imaginary axis, implying a phase margin greater than  $35^\circ$ . The branches of the root locus have been moved into the left half plane as desired, as seen from the new locations of the  $-180^\circ$  arrows.

### CONCLUSIONS

Using pole-zero phase maps helps students to determine the phase of a transfer function from a plot of the poles and zeros. This visualization of the phase of  $L(s)$  helps students develop  $s$ -plane intuition and facilitates the introduction of the analytical tools of classical control, such as Bode plots, Nyquist diagrams, and Evans root-locus plots.

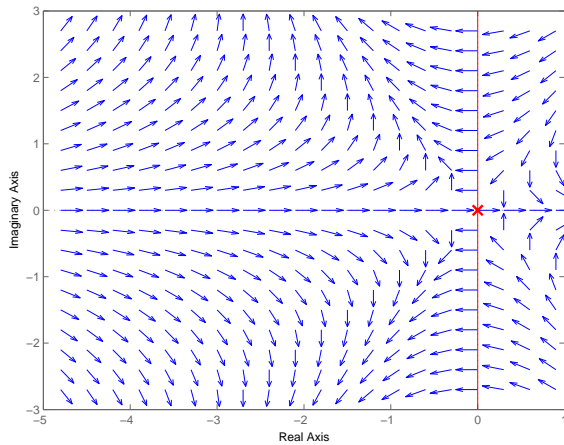


Fig. 7. Pole-zero phase map (blue) and root locus (red) of the double integrator  $L(s) = K/s^2$ . Since the phase along the imaginary axis is always  $-180^\circ$ , the root locus is confined to the imaginary axis, and thus the closed-loop system is unstable for all positive  $K$ .

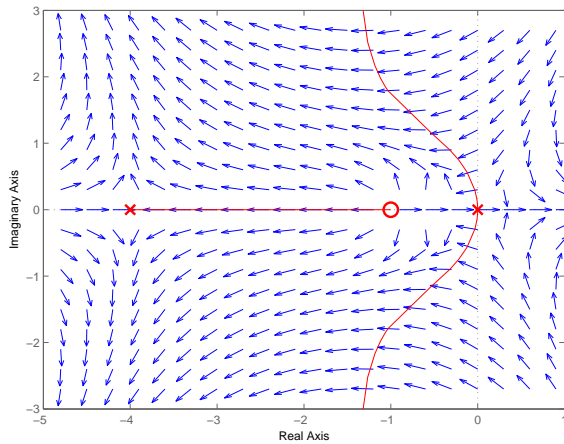


Fig. 8. Pole-zero phase map (blue) and root locus (red) of  $L(s) = K(s + 1)/s^2(0.25s + 1)$ . With lead compensation, the double integrator can be stabilized. The lead zero affects the phase of  $L(j\omega)$  along the positive imaginary axis (improving the phase margin, as desired). This compensation moves the locations of  $-180^\circ$  phase into the left half plane, thus shifting the branches of the root locus into the left half plane.

REFERENCES

[1] W.R. Evans. "Control system synthesis by root locus method." *Trans. AIEE*, vol. 69, pp. 66–69, 1950.  
 [2] W.R. Evans. *Control-System Dynamics*. McGraw-Hill, New York, 1954, pp. 99–100.

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Caveat: The vectors that represent

$$L(s_1) = \frac{s_1 - z}{s_1 - p}$$

are identical to the vectors that represent

$$L(s_1) = -10 \left( \frac{s_1 - z}{s_1 - p} \right).$$

Evans' vector method therefore works only to within a systematic multiplicative factor, which can change the loop-transfer-function magnitude by an arbitrary amount and change the phase by  $180^\circ$ .

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MATLAB commands for Figure 2:

```
[x,y] = meshgrid(-2:0.2:2,-2:0.2:2);
p = angle(1./(x+j*y+1));
u = cos(p); v = sin(p);
u(11,6) = 0; % no arrow at pole
quiver(x,y,u,v,0.7);
axis(2.2*[-1 1 -1 1])
hold on
s = tf('s');
pzmap(1/(s+1),'r')
hold off
title('Pole-Zero Phase Map')
```

**Kent H. Lundberg** attended the Massachusetts Institute of Technology, earning a Ph.D. in electrical engineering in 2002. He is now a lecturer with the Department of Electrical Engineering and Computer Science at MIT. His research interests include the application of control theory to problems in analog circuit design. He is the IEEE Control Systems Magazine Associate Editor for History. He consults for several industrial organizations and collects old textbooks on radar, nuclear energy, and control.