Adaptive Output Feedback Based on Closed-Loop Reference Models

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***Abstract*—This technical note presents the design and analysis of an adaptive controller for a class of linear plants in the presence of output feedback. This controller makes use of a closed-loop reference model as an observer, and guarantees global stability and asymptotic output tracking.**

***Index Terms*—Adaptive control, closed-loop reference model, non-square systems, output-feedback.**

I. Introduction

While adaptive control has been studied since the 60's, the evolution of its use in real systems and the extent to which we fully understand its behavior has only been elucidated within the last decade. Stability of adaptive control systems came only in the 70's, with robustness and extensions to nonlinear systems coming in the 80's and 90's, respectively [1]–[3]. Recent directions in adaptive control pertain to guaranteed transient properties by using a closed-loop architecture for reference models [4]–[11]. In this technical note, we focus on linear *Multi Input Multi Output* (MIMO) adaptive systems with partial state-feedback where we show that such closed-loop reference models can lead to a separation principle based adaptive controller which is simpler to implement compared to the classical ones in [1]–[3]. The simplification comes via the use of reference model states in the construction of the regressor, and not the classic approach where the regressor is constructed from filtered plant inputs and outputs.

In general, the separation principle does not exist for nonlinear systems and few authors have analyzed it. Relevant work on the separation principle in adaptive control can be found in [12], [13]. The structures presented in [12], [13] are very generic, and as such, no global stability results are reported in this literature. Also, due to the generic nature of the results it is *a priori* assumed (or enforced through a saturation function) that the control input and adaptive update law are globally bounded functions with respect to the plant state [13, Assumption 1.2]. No such assumptions are needed in this work and the stability results are global.

The class of MIMO linear plants that we address in this technical note satisfy two main assumptions. The first is that the number of outputs is greater than or equal to the number of inputs, and the second is that the first Markov Parameter has full column rank. The latter is equivalent to a relative degree unity condition in the *Single Input Single Output* (SISO) case. In addition to these two assumptions, the commonly present assumption of stable transmission zeros is needed here as well. With these assumptions, an output feedback adaptive controller is designed that can guarantee stability and asymptotic tracking of the reference output. Unlike [12] and [13], no saturation is needed, and unlike [8]–[10] asymptotic convergence of the tracking error to zero is proved for finite observer gains. Preliminary results on the control scheme presented in this work can be found in [14]. An alternate approach using a linear matrix inequality was developed in [15] and is successfully applied to a hypersonic vehicle model. An analytical approach was developed in [16] to handle a specific class of nonlinear uncertainties and achieves asymptotic convergence of the tracking error to zero with finite observer gains, and is shown to be applicable for a class of flexible aircraft platforms.

The technical note is organized as follows. Section II states the control problem along with our assumptions. Section III proves stability for SISO and square MIMO systems. Section IV analyzes the use of an optimal observer in the design of the closed loop reference model as well as a methodology for extending the design to non-square MIMO systems. Section V contains a simulation example based on the longitudinal dynamics of an aircraft. Conclusions are presented in Section VI.

Notation: The 2-norm for vectors and the induced 2-norm for matrices is denoted as $\|\cdot\|$. The differential operator is defined as $s = d/dt$ throughout. For a real matrix A, the notation A^T is the matrix transpose. We use I to denote the identity matrix. Big O-notation in terms of ν is presented as $O(\nu)$ and unless otherwise stated it is assumed that this holds for ν positive and sufficiently small.

II. Control Problem

The class of plants to be addressed in this technical note is

$$\dot{x} = Ax + B\Lambda u, \qquad y = C^T x \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$. A and Λ are unknown, but B and C are assumed to be known, and only y is assumed to be available for measurement. The goal is to design a control input u so that x tracks the closed-loop reference model state x_m

$$\dot{x}_m = A_m x_m + Br - L(y - y_m), \qquad y_m = C^T x_m \tag{2}$$

where $r \in \mathbb{R}^m$ is the reference input and and L is a feedback gain that will be designed suitably. The reader is referred to references [4]–[7], [17] for its motivation.

The following assumptions are made throughout.

Assumption 1: The product $C^T B$ is full rank.

Assumption 2: The pair $\{A_m, C^T\}$ is observable.

Assumption 3: The system in (1) is minimum phase.

Assumption 4: There exists a $\Theta^* \in \mathbb{R}^{n\times m}$ such that $A + B\Lambda\Theta^{*T} = A_m$ and $K^* \in \mathbb{R}^{m\times m}$ such that $\Lambda K^{*T} = I$.

Assumption 5: Λ is diagonal with positive elements.

Assumption 6: The uncertain matching parameter Θ^*, and the input uncertainty matrix Λ have *a priori* known upper bounds

$$\bar{\theta}^* \triangleq \sup \|\Theta^*\| \text{ and } \bar{\lambda} \triangleq \sup \|\Lambda\|. \tag{3}$$

Assumption 1 corresponds to one of the main assumptions mentioned in the introduction, and that is that the first Markov Parameter is

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nonsingular. The system in (1) is square and therefore the other main assumption mentioned in the introduction is implicitly satisfied. The extension to non-square systems is presented later in the text. Assumption 2 is necessary as our result requires the use of an observer like gain in the reference model, notice the L in (2). Assumption 3 is common in adaptive systems as the KYP Lemma does not hold for plants with a right half plane transmission zero.

Assumptions 4 and 5 imply that the pair $\{A, B\}$ is controllable, and are such that a matching condition is satisfied. Such an assumption is commonly made in plants where states are accessible [1], but is introduced in this problem when only certain outputs are accessible. One application area where such an assumption is routinely satisfied is in the area of aircraft control [10]. Extensions of Assumption 4 to the case when the underlying regressor vector is globally Lipschitz are possible as well [10]. Assumption 5 can be relaxed to Λ symmetric and full rank. Assumption 6 facilitates an appropriate choice of L. The specifics of the control design are now addressed.

For the plant in (1) and (2) satisfying the six assumptions above, we propose the following adaptive controller:

$$u = \Theta^T(t)x_m + K^T(t)r \tag{4}$$

$$\begin{aligned}\dot{\Theta} &= -\Gamma_\theta x_m e_y^T M\\ \dot{K} &= -\Gamma_k r e_y^T M\end{aligned} \tag{5}$$

where

$$M \triangleq C^T B \tag{6}$$

and where $e_y = y - y_m$ and Γ_θ, Γ_k are both positive diagonal free design matrices. The matrix M is referred to as the *mixing matrix* throughout.

The reason for the choice of the control input in (4) is simply because x is not available for measurement, and the reference model state x_m serves as an observer-state. Historically, the use of such an observer has always proved to be quite difficult, as the non-availability of the state proves to be a significant obstacle in determining a stable adaptive law. In the following, it is shown that these obstacles can be overcome for the specific class of multivariable plants that satisfy Assumptions 1 through 6.

From (1), (2), and (4), it is easy to show that the state error $e = x - x_m$ satisfies the dynamics

$$\begin{aligned}\dot{e} &= (A_m + LC^T)e + B\Lambda(\tilde{\Theta}^T x_m + \tilde{K}^T r - \Theta^{*T} e)\\ e_y &= C^T e.\end{aligned} \tag{7}$$

The structure of (7) and the adaptive laws suggest the use of the following Lyapunov function:

$$V = e^T P e + \mathrm{Tr}\left(\Lambda\tilde{\Theta}^T\Gamma_\theta^{-1}\tilde{\Theta}\right) + \mathrm{Tr}\left(\Lambda\tilde{K}^T\Gamma_k^{-1}\tilde{K}\right) \tag{8}$$

where for now it is assumed that $P = P^T > 0$ satisfies the following equation

$$\begin{aligned}(A_m + LC^T)^T P + P(A_m + LC^T) &= -Q\\ PB &= CM\end{aligned} \tag{9}$$

where $Q = Q^T > 0$. Taking the derivative of (8) and using (5), (7), and (9) it can be shown that

$$\dot{V} = -e^T Q e + 2e^T PB\Lambda\Theta^{*T} e. \tag{10}$$

Establishing sign-definiteness of $\dot{V}$ is therefore non-trivial as the size of the sign-indefinite term in (10) is directly proportional to the parametric uncertainty Θ^*, and P and Q are necessarily correlated by (9). In what follows, we will show how L and M can be chosen such that a P and Q satisfying (9) exist and furthermore, $\lim_{t\to\infty} e(t) = 0$. It will be shown that stability for the above adaptive system can only be insured if $Q > 0$ is sufficiently weighted along the CC^T direction.

III. Stability Analysis

A. Stability in the SISO Case

The choice of L is determined in two steps. First, an observer gain L_s and mixing matrix M are selected so that the transfer function $M^T C^T (sI - A - L_s C^T)^{-1} B$ is *Strict Positive Real* (SPR) [1, Definition 2.7].[1] Then the full observer gain L is defined.

Lemma 1: For a SISO $(m = 1)$ system in (1) satisfying Assumptions 1–3 there exists an L_s such that

$$C^T (sI - A_m - L_s C^T)^{-1} B = \frac{a}{s+\rho} \tag{11}$$

where $\rho > 0$ is arbitrary and $a = C^T B$.

Proof: Given that $C^T B$ is non-zero $C^T(sI - A_m - L_s C^T)^{-1} B$ is a relative degree one transfer function. In order to see this fact, consider a system in control canonical form, and compute the coefficient for s^{n-1} in the numerator. By Assumption 2, all zeros of the transfer function $C^T(sI - A)^{-1}B$ are stable, and since zeros are invariant under feedback, $C^T(sI - A_m)^{-1}B$ is minimum phase as well. Assumption 2 implies that the eigenvalues of $A_m + L_s C^T$ can be chosen arbitrarily. Therefore, one can place $n - 1$ of the eigenvalues of $A_m + L_s C^T$ at the $n - 1$ zeros of $C^T(sI - A_m)^{-1}B$ and its n-th eigenvalue clearly at $-\rho$. □

The choice of L_s in Lemma 1 results in a relative degree one transfer function with a single pole not canceling the zeros. This system however need not be SPR as a may be negative; however, $((a^2)/(s + \rho))$ is SPR and thus the following Corollary holds by the KYP Lemma [1, Lemma 2.5].

Corollary 1: If L_s is chosen as in (11) and M selected as in (6), the SISO transfer function $M^T C^T (sI - A_m - L_s C^T)^{-1} B$ is SPR. Therefore, there exists $P = P^T > 0$ and $Q_s = Q_s^T > 0$ such that

$$\begin{aligned}(A_m + L_s C^T)^T P + P(A_m + L_s C^T) &= -Q_s\\ PB &= CM.\end{aligned} \tag{12}$$

Lemma 2: Choosing $L = L_s - \rho BM^T$ where L_s is defined in Lemma 1 and $\rho > 0$ is arbitrary, the transfer function $M^T C^T (sI - A_m - LC^T)^{-1} B$ is SPR and satisfies

$$\begin{aligned}(A_m + LC^T)^T P + P(A_m + LC^T) &= -Q\\ Q &\triangleq Q_s + 2\rho CMM^T C^T\end{aligned} \tag{13}$$

where P and Q_s are defined in (12) and M is defined in (6).

Proof: Starting with the first equation in (12) and adding the term $-\rho(PBM^T C^T + CMB^T P)$ on both sides of the inequality results in the following equality:

$$\begin{aligned}(A_m + LC^T)^T P &+ P(A_m + LC^T)\\ &= -Q_s - \rho(PBM^T C^T + CMB^T P).\end{aligned}$$

Using the second equality in (12) the above equality simplifies to (13) □

Theorem 1: The closed-loop adaptive system specified by (1), (2), (4) and (5), satisfying assumptions 1 to 6, with L as in Lemma 2, M chosen as in (6), and $\rho > \rho^*$ has globally bounded solutions with $\lim_{t\to\infty} e(t) = 0$ with

$$\rho^* = \frac{\bar{\lambda}^2 \bar{\theta}^{*2}}{2\lambda_{\min}(Q_s)} \tag{14}$$

where $\bar{\lambda}$ and $\bar{\theta}^*$ are *a priori* known bounds defined in (3).

Proof: We choose the Lyapunov candidate (8) where P is the solution to (12) and satisfies (13). Taking the time derivative of (8)

[1] M is denoted the mixing matrix, as it mixes the outputs of $C^T(sI - A - L_s C^T)^{-1}B$ so as to achieve strict positive realness.

along the system trajectories in (7), and using the relations in (12), (13), and (5), the following holds:

$$\begin{aligned}\dot{V} = & -e^T(Q+2\rho CMM^TC^T)e - 2e^TPB\Lambda\Theta^{*T}e \\ & + 2e^TPB\Lambda\tilde{\Theta}^Tx_m + 2\mathrm{Tr}\left(\Lambda\tilde{\Theta}^Tx_me_y^TM\right) \\ & + 2e^TPB\Lambda\tilde{K}^Tr + 2\mathrm{Tr}\left(\Lambda\tilde{K}^Tre_y^TM\right)\end{aligned} \tag{15}$$

Using the fact that $PB = CM$ from (12) and the fact the Trace operator is invariant under cyclic permutations the inequality in (15) can be rewritten as

$$\begin{aligned}\dot{V} = & -e^T(Q+2\rho CMM^TC^T)e - 2e^TCM\Lambda\Theta^{*T}e \\ & + 2e^TCM\Lambda\tilde{\Theta}^Tx_m - 2e_y^TM\Lambda\tilde{\Theta}^Tx_m \\ & + 2e^TCM\Lambda\tilde{K}^Tr - 2e_y^TM\Lambda\tilde{K}^Tr.\end{aligned} \tag{16}$$

Using the fact that $e_y = C^Te$, the second and third lines in the above equation equal zero. Therefore, (16) can be written as $\dot{V} = -\mathcal{E}^T\mathcal{Q}(\rho)\mathcal{E}$ where

$$\mathcal{Q}(\rho) = \begin{bmatrix} 2\rho MM^T & M\Lambda\Theta^{*T} \\ \Theta^*\Lambda M^T & Q_s \end{bmatrix} \quad \mathcal{E} = \begin{bmatrix} e_y \\ e. \end{bmatrix}$$

Given that $\rho > \rho^* > 0$, $2M\rho M^T - M\Lambda\Theta^{*T}Q_s^{-1}\Theta^*\Lambda M^T > 0$ by (14) and Q_s is posititve definite by design. By Schur complement, $\mathcal{Q}(\rho)$ is positive definite. Therefore $\dot{V} \leq 0$ and thus $e_y, e, \tilde{\Theta}, \tilde{K} \in \mathcal{L}_\infty$. From (2) it follows that $x_m \in \mathcal{L}_\infty$. From (7) it follows that $\dot{e} \in \mathcal{L}_\infty$. Furthermore, given that $\mathcal{Q}$ is positive definite $e \in \mathcal{L}_2$. Finally, given that $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e} \in \mathcal{L}_\infty$ it follows from Barbalat Lemma that $\lim_{t\to\infty} e(t) = 0$ [1, Corollary 2.9]. □

Remark 1: Theorem 1 implies that a controller as in (4) with the state replaced by the observer state x_m will guarantee stability, thereby illustrating that the separation principle based adaptive control design can be satisfactorily deployed. It should be noted however that two key parameters L and M had to be suitably chosen. If $L = L_s$ then stability is not guaranteed. That is, simply satisfying an SPR condition is not sufficient for stability to hold. It is imperative that Q be chosen as in (13), i.e. be sufficiently positive along the output direction CC^T so as to contend with the sign indefinite term $2e^TPB\Lambda\Theta^{*T}e$ in $\dot{V}$. The result does not require that L_s be chosen so that perfect pole zero cancellation occurs in Lemma 1, all that is necessary is that the phase lead or lag of $C^T(j\omega I - A_m - L_sC^T)^{-1}B$ never exceeds 90 degrees. Finally, it should be noted that any finite $\rho > \rho^*$ ensures stability.

B. Stability in the MIMO Case

Stability in the MIMO case follows the same set of steps as in the SISO case. First, an L_s and M are defined such that the transfer function $M^TC^T(sI - A_m - L_sC^T)^{-1}B$ is SPR. Then L is defined such that the underlying adaptive system is stable. The following Lemmas and Theorem mirror the results from Corollary 1, Lemma 2, and Theorem 1.

Lemma 3: For the MIMO system in (1) satisfying Assumptions 1–3 with M chosen as in (6) there always exists an L_s such that $M^TC^T(sI - A_m - L_sC^T)^{-1}B$ is SPR.

Proof: An algorithm for the existence and selection of such an L_s is given in [18]. □

Remark 2: In order to apply the results from [18], the MIMO system of interest must be 1) minimum phase and 2) M^TC^TB must be symmetric positive definite. By Assumption 3, $C^T(sI - A)^{-1}B$ is minimum phase, and therefore $C^T(sI - A_m)^{-1}B$ is minimum phase as well. Also, given that M is full rank, the transmission zeros of $C^T(sI - A_m)^{-1}B$ are equivalent to the transmission zeros of $M^TC^T(sI - A_m)^{-1}B$. Therefore, condition 1 of this remark is satisfied. We now move on to condition 2. By Assumption 1 C^TB is full rank, and by the definition of M in (6) it follows that $M^TC^TB = B^TCM > 0$, which is a necessary condition for $MC^T(sI - A_m)^{-1}B$ to be SPR [19, Lemma 3]. A similar explicit construction of an L_s such that $M^TC^T(sI - A_m - L_sC^T)^{-1}B$ is SPR can be found in [19].

Lemma 4: Choosing $L = L_s - \rho BM^T$ where L_s is defined in Lemma 3 and $\rho > 0$ is arbitrary, the transfer function $M^TC^T(sI - A_m - LC^T)^{-1}B$ is SPR and satisfies

$$\begin{aligned}(A_m + LC^T)^TP + P(A_m + LC^T) &= -Q \\ Q &\triangleq Q_s + 2\rho CMM^TC^T \\ PB &= CM\end{aligned} \tag{17}$$

where $P = P^T > 0$ and $Q_s = Q_s^T > 0$ are independent of ρ and M is defined in (6).

Theorem 2: The closed-loop adaptive system specified by (1), (2), (4) and (5), satisfying assumptions 1 to 6, with L as in Lemma 4, M chosen as in (6), and $\rho > \rho^*$ has globally bounded solutions with $\lim_{t\to\infty} e(t) = 0$ where ρ^* is defined in (14).

The proofs of Lemma 4 and Theorem 2 follow the same steps as in the proof of Lemma 2 and Theorem 1, respectively.

IV. Extensions

In the previous section a method was presented for choosing L in (2) and M in (5) so that the overall adaptive system is stable and $\lim_{t\to\infty} e(t) = 0$. For the SISO and MIMO cases the proposed method, thus far, is a two step process. First a feedback gain and mixing matrix are chosen such that a specific transfer function is SPR. Then, the feedback gain in the first step is augmented with an additional feedback term of sufficient magnitude along the direction BM^T so that stability of the underlying adaptive system can be guaranteed.

In this section, the method is extended to two different cases. In the first case, we apply this method to an LQG/LTR approach proposed in [10] and show that asymptotic stability can be derived thereby extending the results of [10]. In the second case, the method is extended to non-square plants.

A. MIMO LQG/LTR

The authors in [10] suggested using an LQG approach for the selections of L and M, motivated by the fact the underlying observer (which coincides with the closed-loop reference model as shown in (2)) readily permits the use of such an approach and makes the design more in line with the classical optimal control approach.

In [10] the proposed method is only shown to be stable for finite L, where as in this section it is shown that in fact $\lim_{t\to\infty} e(t) = 0$. Furthermore, we note that the prescribed degree of stability as suggested in [10, Eq. 14.26] through the selection of η is in fact not needed. The analysis below shows that stability is guaranteed due to sufficient weighting of the underlying Q matrix along the CC^T direction.

Let L in (2) be chosen as [10]

$$L = L_\nu \triangleq -P_\nu CR_\nu^{-1} \tag{18}$$

where P_ν is the solution to the Riccati Equation

$$P_\nu A_m^T + A_mP_\nu - P_\nu CR_\nu^{-1}C^TP_\nu + Q_\nu = 0 \tag{19}$$

where $Q_0 = Q_0^T > 0$ in $\mathbb{R}^n$ and $R_0 = R_0^T > 0$ in $\mathbb{R}^m$ and $\nu > 0$, with $Q_\nu = Q_0 + (1 + (1/\nu))BB^T$ and $R_\nu = ((\nu)/(\nu + 1))R_0$. Note that (19) can also be represented as

$$A_\nu^T\tilde{P}_\nu + \tilde{P}_\nu A_\nu = -CR_\nu^{-1}C^T - \tilde{Q}_\nu \tag{20}$$

where $A_\nu = A_m + L_\nu C^T$, $\tilde{P}_\nu = P_\nu^{-1}$ and $\tilde{Q}_\nu = \tilde{P}_\nu Q_\nu\tilde{P}_\nu$. Given that our system is observable and Q and R are symmetric and positive definite, the Riccati equation has a solution P_ν for all fixed ν. We are particularly interested in the limiting solution when ν tends to zero.

The Riccati equation in (19) is very similar to those studied in the LTR literature, with one very significant difference. In LTR methods the state weighting matrix is independent of ν where as in our application Q_ν tends to infinity for small ν.

Lemma 5: If Assumptions 1 through 5 are satisfied, then $\lim_{\nu\to 0}\nu P_\nu = 0$, $\lim_{\nu\to 0} P_\nu = P_0$ where $0 < P_0^T = P_0 < \infty$, and the following asymptotic relation holds:

$$P_\nu = P_0 + P_1\nu + O(\nu^2). \tag{21}$$

Furthermore, there exists a unitary matrix $W \in \mathbb{R}^{m\times m}$ such that

$$P_0 C = BW^T\sqrt{R_0}, \quad \text{and} \quad \tilde{P}_0 B = CR_0^{-1/2}W \tag{22}$$

where $\tilde{P}_0 = P_0^{-1}$ and $W = (UV)^T$ with $B^T CR_0^{-1/2} = U\Sigma V$. Finally, the inverse $\tilde{P}_\nu \triangleq P_\nu^{-1}$ is well defined in limit of small ν and

$$\tilde{P}_\nu = \tilde{P}_0 + \tilde{P}_1\nu + O(\nu^2). \tag{23}$$

A full proof of this result is omitted to save space. The following two facts: 1) $\lim_{\nu\to 0}\nu P_\nu = 0$ and 2) $\lim_{\nu\to 0} P_\nu = P_0$, where $0 < P_0^T = P_0 < \infty$ follow by analyzing the integral cost

$$x^T(0)P_\nu x(0) = \min \int_0^\infty x^T(\tau)Q_\nu x(\tau) + u^T(\tau)R_\nu u(\tau)\, d\tau$$

in the same spirit as was done in [20]. In order to apply the results from [20] the system must be observable (Assumption 2), controllable (Assumptions 4 and 5), minimum phase (Assumption 3), and $C^T B$ must be full rank (Assumption 1). For a detailed analysis of the asymptotic expansions $P_\nu = P_0 + P_1\nu + O(\nu^2)$ and $\tilde{P}_\nu = \tilde{P}_0 + \tilde{P}_1\nu + O(\nu^2)$ see [10, Section 13.3, Theorem 13.2, Corollary 13.1].

The update law for the adaptive parameters is then given as

$$\begin{aligned} \dot{\Theta} &= -\Gamma_\theta x_m e_y^T R_0^{-1/2} W \\ \dot{K} &= -\Gamma_k r e_y^T R_0^{-1/2} W \end{aligned} \tag{24}$$

where W is defined just below (22).

Theorem 3: The closed-loop adaptive system specified by (1), (2), (4), and (24), satisfying assumptions 1 to 6, with L as in (18), and ν sufficiently small has globally bounded solutions with $\lim_{t\to\infty} e(t) = 0$.

Proof: Consider the Lyapunov candidate $V = e^T\tilde{P}_0 e + \mathrm{Tr}(\Lambda\tilde{\Theta}^T\Gamma_\theta^{-1}\tilde{\Theta}) + \mathrm{Tr}(\Lambda\tilde{K}^T\Gamma_k^{-1}\tilde{K})$. Taking the derivative along the system trajectories and substitution of the update laws in (24) results in

$$\begin{aligned} \dot{V} =\, & e^T A_\nu^T \tilde{P}_0 e + e^T\tilde{P}_0 A_\nu e - 2e^T\tilde{P}_0 B\Lambda\Theta^{*T} e \\ & + 2e^T\tilde{P}_0 B\Lambda\tilde{\Theta}^T x_m + 2\mathrm{Tr}\left(\Lambda\tilde{\Theta}^T x_m e_y^T R_0^{-1/2}W\right) \\ & + 2e^T\tilde{P}_0 B\Lambda\tilde{K}^T r + 2\mathrm{Tr}\left(\Lambda\tilde{K}^T r e_y^T R_o^{-1/2}W\right). \end{aligned} \tag{25}$$

The first step in the analysis of the above expression is to replace the elements $A_\nu^T\tilde{P}_0$ and $\tilde{P}_0 A_\nu$ with bounds in terms of $A_\nu^T\tilde{P}_\nu$ and $\tilde{P}_\nu A_\nu$. First note that the following expansions hold in the limit of small ν:

$$\begin{aligned} A_\nu^T\tilde{P}_\nu &= A_\nu^T\tilde{P}_0 + \nu A_\nu^T\tilde{P}_1 + O(\nu) \\ \tilde{P}_\nu A_\nu &= \tilde{P}_0 A_\nu + \nu\tilde{P}_1 A_\nu + O(\nu) \end{aligned}$$

where we have simply expanded the term $\tilde{P}_\nu$. Expanding A_ν as $A_m - P_\nu CR_0^{-1}C^T((\nu+1)/(\nu))$, the above relation simplifies to the following asymptotic relation as ν approaches 0

$$\begin{aligned} A_\nu^T\tilde{P}_\nu &= A_\nu^T\tilde{P}_0 - CR_0^{-1}C^T P_\nu\tilde{P}_1 + O(\nu) \\ \tilde{P}_\nu A_\nu &= \tilde{P}_0 A_\nu - \tilde{P}_1 P_\nu CR_0^{-1}C^T + O(\nu). \end{aligned} \tag{26}$$

Substitution of (26) for the expressions $A_\nu^T\tilde{P}_0$ and $\tilde{P}_0 A_\nu$ in (25) results in the following inequality:

$$\begin{aligned} \dot{V} \le\, & e^T A_\nu^T\tilde{P}_\nu e + e^T\tilde{P}_\nu A_\nu e - 2e^T\tilde{P}_0 B\Lambda\Theta^{*T} e \\ & + e^T CR_0^{-1}C^T P_\nu\tilde{P}_1 e + e^T\tilde{P}_1 P_\nu CR_0^{-1}C^T e + O(\nu)e^T e \\ & + 2e^T\tilde{P}_0 B\Lambda\tilde{\Theta}^T x_m + 2\mathrm{Tr}\left(\Lambda\tilde{\Theta}^T x_m e_y^T R_0^{-1/2}W\right) \\ & + 2e^T\tilde{P}_0 B\Lambda\tilde{K}^T r + 2\mathrm{Tr}\left(\Lambda\tilde{K}^T r e_y^T R_o^{-1/2}W\right). \end{aligned} \tag{27}$$

Substitution of (20) in to the first line above, and using the fact that $\tilde{P}_0 B = CR_0^{-1/2}W$ for the expressions in the bottom two lines

$$\begin{aligned} \dot{V} \le & -e^T\tilde{Q}_\nu e - \frac{\nu+1}{\nu}e_y^T R_0^{-1}e_y + O(\nu)e^T e \\ & + e^T CR_0^{-1}C^T P_\nu\tilde{P}_1 e + e^T\tilde{P}_1 P_\nu CR_0^{-1}C^T e \\ & - 2e^T CR_0^{-1/2}W\Lambda\Theta^{*T} e. \end{aligned}$$

Expanding P_ν, and using the fact that $e_y = C^T e$ and $\nu + 1 \ge 1$, the following inequality holds for ν sufficiently small

$$\begin{aligned} \dot{V} \le & -e^T\tilde{Q}_\nu e - \frac{1}{\nu}e_y^T R_0^{-1}e_y + O(\nu)e^T e \\ & + e_y^T R_0^{-1}C^T P_0\tilde{P}_1 e + e^T\tilde{P}_1 P_0 CR_0^{-1}e_y \\ & - 2e_y^T R_0^{-1/2}W\Theta^{*T} e. \end{aligned} \tag{28}$$

Let $P_\Theta \triangleq -R_0^{-1}C^T P_0\tilde{P}_1 + R_0^{-1/2}W\Theta^{*T}$, then the above inequality can be simplified as $\dot{V} \le -\mathcal{E}^T\mathcal{Q}(\nu)\mathcal{E} + O(\nu)e^T e$ where

$$\mathcal{Q}(\nu) = \begin{bmatrix} \frac{1}{\nu}R_0^{-1} & P_\Theta \\ P_\Theta^T & \tilde{Q}_\nu \end{bmatrix} \quad \text{and} \quad \mathcal{E} = \begin{bmatrix} e_y \\ e \end{bmatrix}. \tag{29}$$

Note that P_Θ is independent of ν and $\lim_{\nu\to 0}\tilde{Q}_\nu \ge \tilde{P}_0 Q_0\tilde{P}_0 > 0$. Thus for ν sufficiently small $(1/\nu)R_0^{-1} - P_\Theta\tilde{Q}_\nu^{-1}P_\Theta^T > 0$. Therefore $\mathcal{Q}(\nu)$ is positive definite and for ν sufficiently small $\mathcal{Q}(\nu) - O(\nu)I > 0$ as well, where I is the identity matrix. Thus the adaptive system is bounded for sufficiently small ν. As before, it follows that $e \in \mathcal{L}_2$, and by Barbalat Lemma, $\lim_{t\to\infty} e(t) = 0$. □

Remark 3: The same discussion for the SISO and MIMO cases is valid for the LQG/LTR based selection of L. Stability follows do to the fact that the Lyapunov candidate suitably includes the "fast dynamics" along the e_y error dynamics. This fact is illustrated in (20) with the term $CR_\nu^{-1}C^T$ appearing on the right hand, which when expanded in terms of ν takes the form $((1+\nu)/(\nu))CR_0^{-1}C^T$. By directly comparing $((1+\nu)/(\nu))CR_0^{-1}C^T$ to the term $2\rho CMM^T C^T$ on the right hand side of (13), increasing ρ and decreasing ν have the same affect on the underlying Lyapunov equations. Thus, stability is guaranteed so long as ρ is sufficiently large or equivalently, ν sufficiently small.

B. Extension to Non-Square Systems

Consider dynamics of the following form:

$$\dot{x} = Ax + B_1\Lambda u, \qquad y = C^T x \tag{30}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $y\in\mathbb{R}^p$ and $p > m$. $B_1\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{n\times p}$ are known. $A\in\mathbb{R}^{n\times n}$ and $\Lambda\in\mathbb{R}^{m\times m}$ are unknown. To address the non-square aspect Assumption 1 is replaced with the following:

Assumption 7: $\mathrm{Rank}(C) = p$ and $\mathrm{Rank}(C^T B_1) = m$.

Again, the goal is to design a controller such that $x(t)$ follows the reference model:

$$\dot{x}_m = A_m x_m + B_1 r - Le_y, \qquad y_m = C^T x_m \tag{31}$$

where $C^T(sI - A_m)^{-1}B_1$ represents the ideal behavior responding to a command r.

Lemma 6: For a non-square system in the form of (30) and (31) that satisfies Assumptions 2, 3, and 7, there exists a $B_2 \in \mathbb{R}^{n\times(p-m)}$ such that the "squared-up" system $C^T(sI - A_m)^{-1}B$ is minimum phase, and $C^T B$ is full rank, where

$$B = [\,B_1 \quad B_2\,]. \tag{32}$$

Proof: The reader is referred to [21] for further details. □

We now consider the squared-up plant $\{A_m, B, C^T\}$ and state the lemmas corresponding to Lemma 3 and Lemma 4.

Lemma 7: For the MIMO system in (30) satisfying Assumptions 2, 3 and 7 with M chosen as in (6) with B as defined in (32) there exists an L_s such that $M^T C^T (sI - A_m - L_s C^T)^{-1} B$ is SPR.

Lemma 8: Choosing $L = L_s - \rho B M^T$ where L_s is defined in Lemma 3 and $\rho > 0$ is arbitrary, the transfer function $M^T C^T (sI - A_m - LC^T)^{-1} B$ is SPR and satisfies

$$\begin{aligned}(A_m + LC^T)^T P + P(A_m + LC^T) &= -Q\\ Q &\triangleq Q_s + 2\rho C M M^T C^T\\ PB &= CM\end{aligned} \tag{33}$$

where $P = P^T > 0$ and $Q_s = Q_s^T > 0$ are independent of ρ and M is defined in (6).

We should note that the B matrix above corresponds to additional $p - m$ inputs which are fictitious. The following corollary helps in determining controllers that are implementable.

Corollary 2: Choosing $L = L_s - \rho B M^T$ where L_s is defined in Lemma 7 and $\rho > 0$ is arbitrary, the transfer function $M_1^T C^T (sI - A_m - LC^T)^{-1} B_1$ is SPR and M_1 is defined by the partition $M = [M_1\ M_2]$ which satisfies $P[B_1\ B_2] = C[M_1\ M_2]$.

Accordingly, we propose the following adaptive law:

$$\begin{aligned}\dot{\Theta} &= -\Gamma_\theta x_m e_y^T M_1\\ \dot{K} &= -\Gamma_k r e_y^T M_1.\end{aligned} \tag{34}$$

The following theorem shows that the overall system is globally stable and $\lim_{t\to\infty} e(t) = 0$.

Theorem 4: The closed-loop adaptive system specified by (30), (31), (4), and (34), satisfying assumptions 2 to 7, with B chosen as in (32), L as in Lemma 8, M chosen as in Equation (6), with M_1 defined in Corollary 2, and $\rho > \rho^*$ has globally bounded solutions with $\lim_{t\to\infty} e(t) = 0$, where ρ^* is defined as

$$\rho^* = \frac{\bar{\lambda}^2 \bar{\theta}^{*2} \|M_1\|^2}{2\lambda_{\min}(Q_s)\lambda_{\min}(MM^T)}. \tag{35}$$

Proof: The proof follows as in that of Theorem 1. ■

V. Simulation Study

For the simulation study, we compare the performance of a combined linear and adaptive LQG controller to an LQR controller, which is full states accessible by definition. The uncertain system to be controlled is defined as

$$\dot{x}_p = A_p x_p + B_p u \quad \text{and} \quad y_p = C_y^T x_p$$

where $x_p = [V\ \ \alpha\ \ q\ \ \theta]^T$ is the state vector for the plant consisting of: velocity in ft/s, angle of attack in radians, pitch rate in radians per second, and pitch angle in radians. The control input consists of $u = [T\ \ \delta]^T$, the throttle position percentage and elevator position in degrees. The measured outputs are $y_p = [V\ \ q\ \ h]^T$ where h is height measured in feet. We note that two of the states for this example are not available for measurement, the angle of attack and the pitch angle. The pitch angle is never directly measurable and is always reconstructed from the pitch rate through some filtering process. The angle of attack however is usually available for direct measurement in most classes of aircraft. There are several classes of vehicles however where this information is hard to obtain directly: weapons, munitions, small aircraft, hypersonic vehicles, and very flexible aircraft, just to name a few.

In this example, we intend to control the altitude of the aircraft, and for this reason an integral error is augmented to the plant. The extended state plant is thus defined as

$$\dot{x} = Ax + B_1 u + B_z r \quad \text{and} \quad y = C^T x$$

where $y_z = h$, r is the desired altitude

$$x = \begin{bmatrix} x_p \\ \int(y - r)\end{bmatrix},\quad A = \begin{bmatrix} A_p & 0_{4\times1} \\ C_z & 0_{1\times1}\end{bmatrix},\quad B_1 = \begin{bmatrix} B_p \\ 0_{1\times2}\end{bmatrix}$$
$$B_z = \begin{bmatrix} 0_{4\times1} \\ -I_{1\times1}\end{bmatrix},\quad C^T = \begin{bmatrix} C^T & 0_{3\times1} \\ 0_{1\times4} & I_{1\times1}\end{bmatrix},\quad y = \begin{bmatrix} y_p \\ \int(y_z - r)\end{bmatrix}.$$

The reference system is defined as

$$\dot{x}_m = A_m x_m + B_z r - L_\nu (y - y_m) \quad \text{and} \quad y_m = C^T x_m$$

where $A_m = A_{nom} + B_1 K_R^T$, with $K_R^T = -R_R^{-1} B_p P_R$ the solution to the algebraic Riccati equation

$$A_{nom}^T P_R + P_R A_{nom} - P_R B R_R^{-1} B^T P_R + Q_R = 0$$

and

$$A_{nom} = \begin{bmatrix} A_{p,nom} & 0_{4\times1} \\ C_z & 0_{1\times1}\end{bmatrix}.$$

The closed-loop reference model gain L_ν is defined as in (18) where we have squared up the input matrix through the artificial selection of a matrix B_2 and defined $B = [B_1\ B_2]$ so that $C^T B$ is square, full rank, and $C^T(sI - A_m)^{-1} B$ is minimum phase. The control input for the linear and adaptive LQG controller is defined as

$$u = K_R^T x_m + \Theta^T x_m$$

where the update law for the adaptive parameters is defined as

$$\dot{\Theta} = -\Gamma x_m e_y^T M_1$$

with M_1 the first m colums of $R_0^{-1/2} W$ where W is defined just below (22). The LQR controller is defined as

$$u = K_R^T x.$$

All simulation and design parameters are given in Appendix A. Note that the free design parameter Γ has zero for the last entry, this is due to the fact that for an uncertainty in A_p feedback from the integral error state is not needed for a matching condition to exist. The simulation results are now presented.

Fig. 1 contains the trajectories of the state space for the adaptive controller (black), linear controller (gray), reference model x_m (black dotted), and reference command height (gray dashed). The reference command in height was chosen to be a filtered step, as can be seen by the gray dashed line. The plant when controlled only by the full state linear optimal controller is unable to maintain stability as can be seen by the diverging trajectories. The reference model trajectories are only visibly different from the plant state trajectories under adaptive control in the angle of attack subplot and the pitch angle subplot, the two states which are not measurable. Fig. 2 contains the control input trajectories for the adaptive controller. There are two points to take away from the simulation example. First, the adaptive output feedback controller is able to stabilize the system while the full state accessible linear controller is not. Second, the state trajectories and control inputs exhibit smooth trajectories. This smooth behavior is rigorously justified in [4] for a simpler class of closed-loop reference models.

VI. Conclusion

This technical note presents methods for designing output feedback adaptive controllers for plants that satisfy a states accessible matching condition, thus recovering a separation like principle for this class of adaptive systems, similar to linear plants.

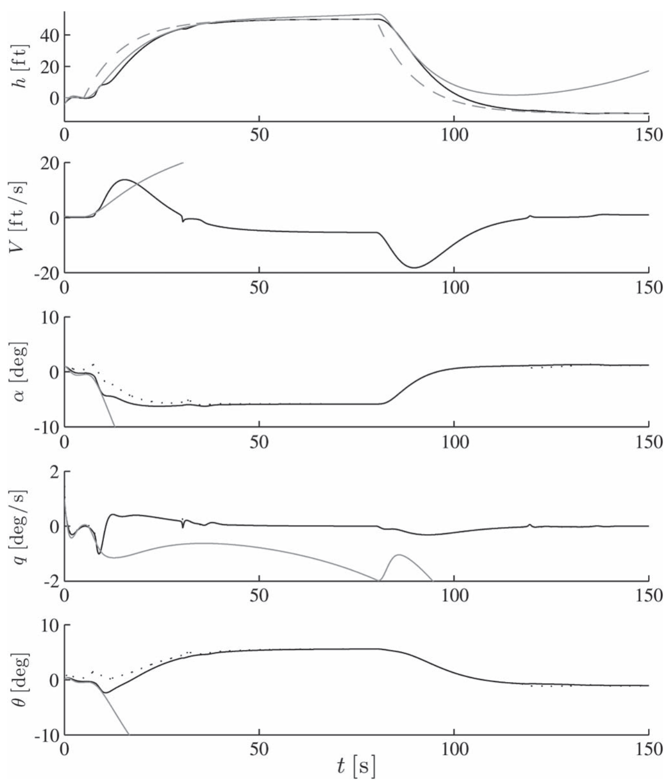

Fig. 1. Trajectories in state space from the adaptive controller (black), linear LQR controller (gray), reference model x_m (black dotted), reference command for height (gray dashed).

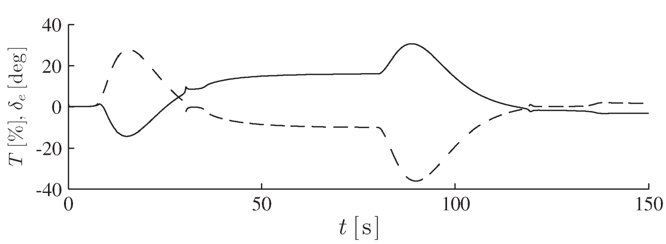

Fig. 2. Control inputs from the adaptive controller, throttle percentage (dashed) and elevator position (solid).

Appendix A

The plant parameters are given as

$$A_{p,nom} = \begin{bmatrix} -0.038 & 18.94 & 0 & -32.174 \\ -0.001 & -0.632 & 1 & 0 \\ 0 & -0.759 & -0.518 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 10.1 & 0 \\ 0 & -0.0086 \\ 0.025 & -0.011 \\ 0 & 0 \end{bmatrix}$$

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -250 & 0 & 250 \end{bmatrix}$$

$$C_z = \begin{bmatrix} 0 & -250 & 0 & 250 \end{bmatrix}$$

$$A_p = A_{p,nom} + B_p \begin{bmatrix} -2 & 1.5 & 2 & -2 \\ 1.5 & -2 & 2 & 1 \end{bmatrix}.$$

The linear control design parameters are given as $Q_R = \mathrm{diag}([1\ \ 1\ \ .1\ \ 0\ \ .1])$ and $R_R = \mathrm{diag}(1\ \ 10])$ where $K_R^T = -R_R^{-1} B_p P_R$ with P_R the solution to the control Riccati equation. The adaptive control design is given by $Q_0 = I_{(n+q)\times(n+q)}$, $R_0 = I_{(p+q)\times(p+q)}$, $\Gamma = \mathrm{diag}([1\ \ 1\ \ 1\ \ 1\ \ 0])$, $\nu = 0.01$, and $B_2^T = \begin{bmatrix} 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}$.

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