Adaptation and Synchronization over a Network: Asymptotic Error Convergence and Pinning

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Outline

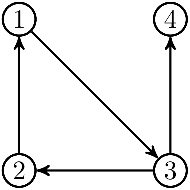
- Graph Notation, Balancing
- Problem Statement
- Undirected Graphs
- Directed Graphs

Graph Notation and Consensus

Graph : $\mathcal{G}(\mathcal{V}, \mathcal{E})$

Vertex Set : $\mathcal{V} = \{1, 2, \dots, n\}$

Edge Set : $(i, j) \in \mathcal{E} \subset \mathcal{V} \times \mathcal{V}$



Adjacency Matrix : $[\mathcal{A}]_{ij} = \begin{cases} 1 & \text{if } (j, i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$

In-degree Laplacian : $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$

In-degree of Node i : $[\mathcal{D}]_{ii}$

Consensus Problem

$$\Sigma_i : \quad \dot{x}_i = -\sum_{j \in \mathcal{N}_{\text{in}}(i)} (x_i - x_j)$$

Using Graph Notation

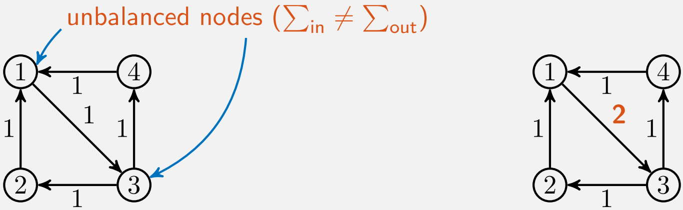
$$\Sigma : \quad \dot{\mathbf{x}} = -\mathcal{L}\mathbf{x}, \qquad \mathbf{x} = [x_1,\ x_2, \dots,\ x_n]^{\mathsf{T}}$$

Graph Balancing (Output Balancing)

Strongly Connected (SC) there is a walk between any two vertices in $\mathcal{G}$.

- $\lambda_1(\mathcal{L}(\mathcal{G})) = 0 > \lambda_2(\mathcal{L}(\mathcal{G})) \geq \cdots \geq \lambda_n(\mathcal{L}(\mathcal{G}))$

Balanced in-degree = out-degree.



- **Output Balancing**: diagonal matrix $D \succ 0$ s.t. weighted graph $\tilde{\mathcal{G}}(\mathcal{V}, \mathcal{E}, D\mathcal{A})$ is balanced.
- Example: $D = \text{diag}([1, 1, 2, 1]^\mathsf{T})$

- $\mathcal{G}$ **SC** $\implies \mathcal{L}(\mathcal{G})\mathbf{1} = \mathbf{0}$ (...holds for any $\mathcal{G}$ not just SC $\mathcal{G}$)
- $\tilde{\mathcal{G}}$ **SC** & **balanced** $\implies \mathbf{1}^\mathsf{T}\tilde{\mathcal{L}}(\tilde{\mathcal{G}}) = \mathbf{0}^\mathsf{T}$

$\mathcal{G}$ **strongly connected** $\exists$ **balancing** D $\implies \mathbf{1}^\mathsf{T} D\mathcal{L}(\mathcal{G}) = \mathbf{0}^\mathsf{T}$

Incomplete Literature on Balancing and Consensus

- [Tsitsiklis, *Phd Thesis* '84] [Jadbabaie, Lin & Morse, *TAC* '03]
    - Focused on discrete time consensus
- [Olfati-Saber & Murray *TAC* '04]
    - Focussed on balanced graphs
- 3 Papers by Chai Wah Wu all in 2005 (IBM Research NY)
    - Focussed on directed graphs that were not balanced
    - Synchronization of nonlinear systems
    - Used the graph balancing matrix in his proofs
- [Makhdoumi & Ozdaglar *CDC* '15]
    - A graph balancing itself through neighbor consensus (distributed subgradient method)

There are lots and lots of papers on this topic, however the existence of graph balancing matrices has not been fully exploited in the adaptive consensus literature.

Problem at Hand

Scalar dynamics on a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$

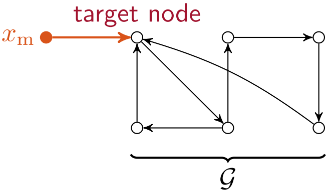
$$\dot{x}_i(t) = \underbrace{a_i}_{\text{unknown}} x_i(t) + u_i(t), \quad i \in \mathcal{V}$$

Control $u_i(t) = \underbrace{\hat{k}_i(t)}_{\text{adaptive}} x_i(t) + \underbrace{\hat{r}_i(t)}_{\text{adaptive}}$

Scalar reference model

$$\dot{x}_{\mathrm{m}}(t) = a_{\mathrm{m}} x_{\mathrm{m}}(t) + r$$

Communication Topology



Target set $\mathcal{T} \subset \mathcal{V}$, nodes that receive information from the reference model.

Goal: Design adaptive laws so that $x_i \to x_{\mathrm{m}}$ while only communicating over $\mathcal{G}$

- But wait ... don't we need to add consensus and pinning to the input

$$u_i = \underbrace{\hat{k}_i(t)}_{\text{adaptive}} x_i + \underbrace{\hat{r}_i(t)}_{\text{adaptive}} - \underbrace{\sum_{j \in \mathcal{N}_{\text{in}}(i)} (x_i - x_j)}_{\text{consensus}} - \underbrace{\sum_{i \in \mathcal{T}} (x_i - x_{\mathrm{m}})}_{\text{pinning}}$$

- ... NOT necessarily

Compact Representation of Dynamics

Error $e_i = x_i - x_m$

$$\dot{\mathbf{e}} = \underbrace{\mathbf{A}_m}_{\text{diag}(\mathbf{1}a_m)} \mathbf{e} + \underbrace{\tilde{\mathbf{K}}}_{\text{diag}(\tilde{\mathbf{k}})} \mathbf{x} + \tilde{\mathbf{r}}$$

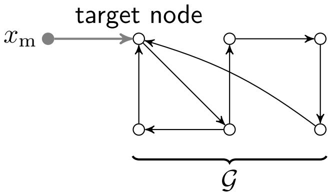
Locally computable errors

$$\text{local error} = \underbrace{\sum_{j \in \mathcal{N}_{\text{in}}(i)} (x_i - x_j)}_{\text{consensus}} - \underbrace{\sum_{i \in \mathcal{T}} (x_i - x_m)}_{\text{pinning}}$$

$$\mathbf{e}_\beta = \underbrace{(\mathcal{L} + \mathcal{M})}_{=:\mathcal{B}} \mathbf{e}$$

$$[\mathcal{M}]_{ii} = \begin{cases} 1 & \text{if } i \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$$

NOTE: $\mathcal{B}$ is full rank & if D is a balancing of the graph, then $\mathcal{B}^\mathsf{T} D + D\mathcal{B} \succ 0$



Target set $\mathcal{T} \subset \mathcal{V}$

Model

$$\dot{x}_i = a_i x_i + u_i$$
$$u_i = \hat{k}_i(t) x_i + \hat{r}_i(t)$$
$$\dot{x}_m = a_m x_m + r$$

Global Parameters

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^\mathsf{T}$$
$$\hat{\mathbf{k}} = [\hat{k}_1, \hat{k}_2, \ldots \hat{k}_n]^\mathsf{T}, \; \hat{\mathbf{r}} \ldots$$
$$\tilde{\mathbf{k}} = \hat{\mathbf{k}} - \mathbf{k}^*, \; \tilde{\mathbf{r}} \ldots$$
$$\mathbf{x}_m = \mathbf{1} x_m$$

Symmetric Graph $\mathcal{G}$

Error Dynamics & Local Error

$$\dot{\mathbf{e}} = \mathbf{A}_{\mathrm{m}}\mathbf{e} + \tilde{\mathbf{K}}\mathbf{x} + \tilde{\mathbf{r}}$$

$$\mathbf{e}_\beta = \underbrace{(\mathcal{L} + \mathcal{M})}_{=:\mathcal{B}}\mathbf{e}$$

Target set $\mathcal{T}$ + Target matrix $\mathcal{M}$

$$[\mathcal{M}]_{ii} = \begin{cases} 1 & \text{if } i \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$$

Adaptive Updates

$$\dot{\hat{\mathbf{k}}} = -\mathrm{diag}(\mathbf{x})\mathbf{e}_\beta$$

$$\dot{\hat{\mathbf{r}}} = -\mathbf{e}_\beta$$

Theorem

For the error dynamics and update laws above, if $\mathcal{G}$ is a **strongly connected symmetric** graph and there is **at least one target node**, then all signals are uniformly bounded and $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$

$$V(\mathbf{e}_\beta, \tilde{\mathbf{k}}, \tilde{\mathbf{r}}) = \mathbf{e}_\beta^{\mathsf{T}}\mathcal{B}^{-1}\mathbf{e}_\beta + \tilde{\mathbf{k}}^{\mathsf{T}}\tilde{\mathbf{k}} + \tilde{\mathbf{r}}^{\mathsf{T}}\tilde{\mathbf{r}}$$

$$\vdots$$

$$\dot{V} = 2a_{\mathrm{m}}\mathbf{e}_\beta^{\mathsf{T}}\mathcal{B}^{-1}\mathbf{e}_\beta.$$

NOTE: $a_{\mathrm{m}} < 0$ □

General Graphs

- For a generic graph $\mathcal{L}$ is **not** symmetric $\implies$ analysis harder.
- Analysis will now require the input to use **consensus & pinning**

$$\mathbf{u} = \text{diag}(\hat{\mathbf{k}}(t))\mathbf{x} + \hat{\mathbf{r}}(t) + c\mathcal{B}\mathbf{e}, \qquad \mathcal{B} = \underbrace{\mathcal{L}}_{\text{consensus}} + \underbrace{\mathcal{M}}_{\text{pinning}}, \quad c < 0$$

- Most of the literature in this area has focussed on symmetric graph
  - Lots of work by Mario di Bernardo is related to this work

A. Das and Frank L. Lewis, Automatica **46** (2010), no. 12,

  - An a-priori bound on the regressor is assumed ($\mathbf{x}$ is a-priori bounded)
  - Does not exploit the fact that D is not only s.t. $\mathcal{L}^T D + D\mathcal{L} \succ 0$ but is **also a graph balancing as well**
  - Error $\mathbf{e}$ converges to compact set proportional to $\|\mathbf{k}^*\|$
  - Adaptive laws use D explicitly (thus all agents must know graph structure or learn D).

- **Main contribution**: a different way of proving stability
  - $\mathbf{e}(t) \to \mathbf{0}$
  - Projection is used ([Das and Lewis] use sigma modification)
  - We exploit the graph balancing condition
  - If time remains I will show the [Das and Lewis] proof.

General Graphs, Theorem Statement

Slightly easier problem for space reasons

$$\mathbf{u} = \text{diag}(\hat{\mathbf{k}}(t))\mathbf{x} + \underbrace{\mathbf{r}}_{\text{known}} + c\mathcal{B}\mathbf{e}$$

$$\dot{\mathbf{e}} = \mathbf{A}_{\mathrm{m}}\mathbf{e} + \tilde{\mathbf{K}}\mathbf{x} + c\mathcal{B}\mathbf{e}$$

$$\dot{\hat{\mathbf{k}}} = \text{proj}_{\infty}(-\text{diag}(\mathbf{x})\mathcal{B}\mathbf{e}, \hat{\mathbf{k}}, k_{\max})$$

Theorem

For the error dynamics and update laws above, with $\mathcal{G}$ a **strongly connected** graph with there being **at least one target node**, and with c **sufficiently negative**, then all signals are uniformly bounded and $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$

$V = \mathbf{e}^{\mathsf{T}} D\mathbf{e}$ D is a graph balancing $\mathbf{1}^{\mathsf{T}} D\mathcal{L} = \mathbf{0}^{\mathsf{T}}$

$$\dot{V} = 2a_{\mathrm{m}}\mathbf{e}^{\mathsf{T}} D\mathbf{e} + \mathbf{e}^{\mathsf{T}} D\tilde{\mathbf{K}}\mathbf{x} + \mathbf{x}^{\mathsf{T}}\tilde{\mathbf{K}}D\mathbf{e} + c\mathbf{e}^{\mathsf{T}}((\mathcal{L}^{\mathsf{T}} + \mathcal{M})D + D(\mathcal{L} + \mathcal{M}))\mathbf{e}$$

$$\dot{V} = 2a_{\mathrm{m}}\mathbf{e}^{\mathsf{T}} D\mathbf{e} + \mathbf{e}^{\mathsf{T}} D\tilde{\mathbf{K}}\mathbf{x} + \mathbf{x}^{\mathsf{T}}\tilde{\mathbf{K}}D\mathbf{e} + c\underbrace{\mathbf{e}^{\mathsf{T}}((\mathcal{L}^{\mathsf{T}} + \mathcal{M})D + D(\mathcal{L} + \mathcal{M}))\mathbf{e}}_{\text{NOTE: } D \text{ is a graph balancing}}$$

General Graphs, Proof continued

$V = \mathbf{e}^\mathsf{T} D \mathbf{e}$ D is a graph balancing $\mathbf{1}^\mathsf{T} D \mathcal{L} = \mathbf{0}^\mathsf{T}$

$$\dot{V} = 2a_\mathrm{m} \mathbf{e}^\mathsf{T} D \mathbf{e} + \mathbf{e}^\mathsf{T} D \tilde{\mathbf{K}} \mathbf{x} + \mathbf{x}^\mathsf{T} \tilde{\mathbf{K}} D \mathbf{e} + c \underbrace{\mathbf{e}^\mathsf{T} ((\mathcal{L}^\mathsf{T} + \mathcal{M}) D + D(\mathcal{L} + \mathcal{M}))\mathbf{e}}_{\text{NOTE: } D \text{ is a graph balancing}}.$$

Recall $\mathbf{e} = \mathbf{x} - \mathbf{1} x_\mathrm{m}$

$$(\underbrace{\mathbf{x} - \mathbf{1} x_\mathrm{m}}_{\mathbf{e}})^\mathsf{T} D \mathcal{L} = \mathbf{x}^\mathsf{T} D \mathcal{L}$$

$$c \underbrace{\mathbf{e}^\mathsf{T} ((\mathcal{L}^\mathsf{T} + \mathcal{M}) D + D(\mathcal{L} + \mathcal{M}))\mathbf{e}}_{\text{separate into two halves}}$$

$$\dot{V} = \begin{bmatrix} \mathbf{e}^\mathsf{T} & \mathbf{x}^\mathsf{T} \end{bmatrix} \begin{bmatrix} 2a_\mathrm{m} D + \frac{c}{2} \underbrace{Q_1}_{\succ 0} & D \underbrace{\tilde{\mathbf{K}}}_{\text{bdd by Proj}} \\ \underbrace{\tilde{\mathbf{K}}}_{\text{bdd by Proj}} D & \frac{c}{2}(\mathcal{L}^T D + D \mathcal{L}) \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{x} \end{bmatrix}$$

- For c sufficiently negative matrix becomes $\prec 0$.
- Analyze in two scenarios $\mathbf{x} \notin \mathbb{R}^n_\mathbf{1} \setminus \mathbf{0}$ and $\mathbf{x} \in \mathbb{R}^n_\mathbf{1} \setminus \mathbf{0}$
- $\mathbf{x}$ can never be in $\mathbb{R}^n_\mathbf{1} \setminus \mathbf{0}$ for any finite amount of time.

□

Compare and Contrast with [Das and Lewis]

Our Result

$$\dot{\hat{\mathbf{k}}} = \text{proj}_\infty(-\text{diag}(\mathbf{x})\mathcal{B}\mathbf{e}, \hat{\mathbf{k}}, k_{\max})$$

$$\dot{V} = \begin{bmatrix}\mathbf{e}^\mathsf{T} & \mathbf{x}^\mathsf{T}\end{bmatrix} \begin{bmatrix} 2a_\mathrm{m}D + \frac{c}{2}\underbrace{Q_1}_{\succ 0} & D\underbrace{\tilde{\mathbf{K}}}_{\text{bdd by Proj}} \\ \underbrace{\tilde{\mathbf{K}}}_{\text{bdd by Proj}}D & \frac{c}{2}(\mathcal{L}^T D + D\mathcal{L}) \end{bmatrix} \begin{bmatrix}\mathbf{e} \\ \mathbf{x}\end{bmatrix}$$

[Das and Lewis]

$$\dot{\hat{\mathbf{k}}} = -D\text{diag}(\mathbf{x})\mathcal{B}\mathbf{e} - \underbrace{\sigma\hat{\mathbf{k}}}_{\text{Sigma mod}}$$

$$\dot{V} = \begin{bmatrix}\mathbf{e}^\mathsf{T} & \tilde{\mathbf{k}}^\mathsf{T}\end{bmatrix} \begin{bmatrix} cq_{11} & q_{12} \\ q_{21} & -\sigma \end{bmatrix} \begin{bmatrix}\mathbf{e} \\ \tilde{\mathbf{k}}\end{bmatrix} + \begin{bmatrix} q_4 & -\sigma \end{bmatrix} \begin{bmatrix}\mathbf{e} \\ \tilde{\mathbf{k}}\end{bmatrix}$$

- Different structure, $\mathbf{x}$ replaced by $\tilde{\mathbf{k}}$ (given by sigma mod)
- For c negative, signals converges to a compact set

General Graphs the Complete Problem

Now r is unknown

$$\mathbf{u} = \mathrm{diag}(\hat{\mathbf{k}}(t))\mathbf{x} + \underbrace{\hat{\mathbf{r}}(t)}_{\text{adaptive}} + c\mathcal{B}\mathbf{e}$$

$$\dot{\mathbf{e}} = \mathbf{A}_{\mathrm{m}}\mathbf{e} + \tilde{\mathbf{K}}\mathbf{x} + \tilde{\mathbf{r}} + c\mathcal{B}\mathbf{e}$$

$$\dot{\hat{\mathbf{k}}} = \mathrm{proj}_{\infty}(-\mathrm{diag}(\mathbf{x})\mathcal{B}\mathbf{e}, \hat{\mathbf{k}}, k_{\max})$$

$$\dot{\hat{\mathbf{r}}} = -\mathcal{B}\mathbf{e} - \underbrace{\mathcal{L}\hat{\mathbf{r}}}_{\text{sharing estimates}}$$

Theorem

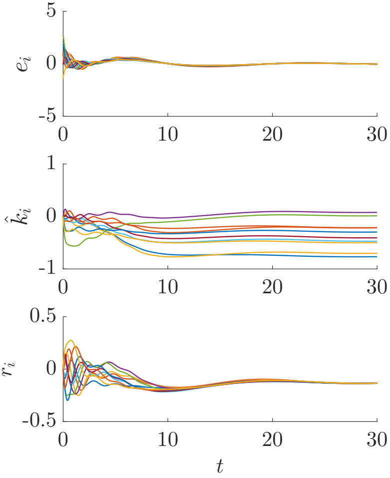
For the error dynamics and update laws above, with $\mathcal{G}$ a **strongly connected** graph with there being **at least one target node**, and with $\boldsymbol{c}$ **sufficiently negative**, then all signals are uniformly bounded and $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$

$$V = \mathbf{e}^{\mathsf{T}} D\mathbf{e} + \tilde{\mathbf{r}}^{\mathsf{T}} D\tilde{\mathbf{r}}.$$

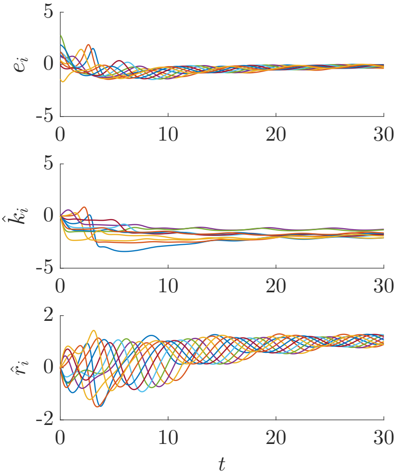
Simulations

Graph is Directed Cycle of 10 nodes

$$\mathbf{u} = \mathsf{diag}(\hat{\mathbf{k}}(t))\mathbf{x} + \hat{\mathbf{r}}(t) - 5\mathcal{B}\mathbf{e}$$

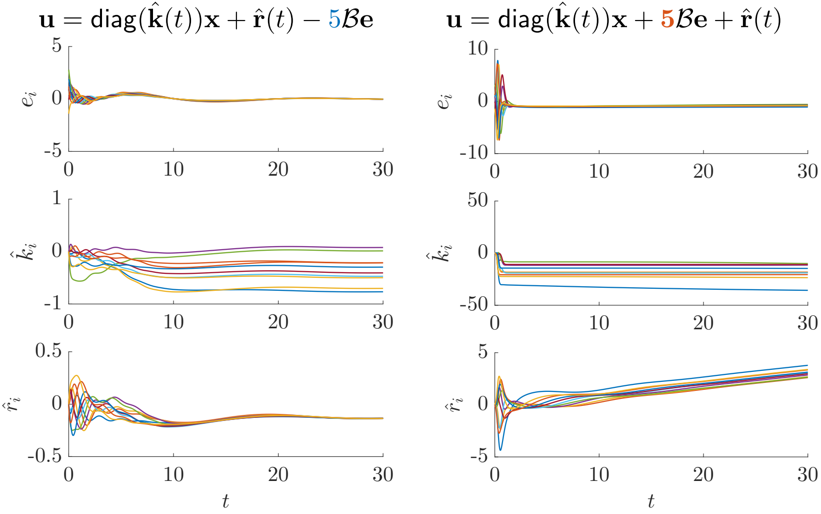


$$\mathbf{u} = \mathsf{diag}(\hat{\mathbf{k}}(t))\mathbf{x} + \hat{\mathbf{r}}(t)$$



Simulations Cont.

What about desynchronous inputs?



Summary

- For symmetric graphs consensus can be achieved with only local adaptive control
- For the directed case we proved asymptotic convergence $\mathbf{e} \to \mathbf{0}$
- **Conjecture**: for digraphs consensus can be achieved with only local adaptive control
- If you are interested in understanding the desynchronous case come to my second talk in this same session

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