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Comments on "Design of Strictly Positive Real Systems Using Constant Output Feedback"

Itzhak Barkana

***Abstract*—A recent publication states and proves the conditions under which a linear time-invariant system, with state-space realization A, B, C, can be made strictly positive real via constant output feedback. This note is intended to briefly present the development of the proof and to give due credit to the first proofs of this statement.**

***Index Terms*—Adaptive control, almost strictly positive real (ASPR), positive real systems, stability.**

I. INTRODUCTION

In [1, Th. 1], Huang *et al.* prove the following result.

Theorem 1: Any strictly proper system $\{A, B, C\}$, where $A \in \mathbf{R}^{n,n}, B \in \mathbf{R}^{n,m}, C \in \mathbf{R}^{m,n}$, with the $m * m$ transfer function $T(s) = C(sI - A)^{-1}B$, and with CB positive–definite symmetric can be made strictly positive real (SPR) via constant output feedback if there exists a positive–definite matrix Po such that $M^T(N^T PNA + A^T N^T PN)M < 0$. Here, M and N are such that $CM = 0$ and $NB = 0$.

The interested reader may wonder what classes of systems satisfy the conditions of the theorem. Therefore, it will be shown here that this theorem is equivalent to the following more direct formulation.

Theorem 2: Any strictly proper strictly minimum-phase system $\{A, B, C\}$, where $A \in \mathbf{R}^{n,n}, B \in \mathbf{R}^{n,m}, C \in \mathbf{R}^{m,n}$, with the $m * m$ transfer function $T(s) = C(sI - A)^{-1}B$, and with CB positive–definite symmetric can be made strictly positive real via constant output feedback.

Theorem 2 of this note thus completes and clarifies the implications of the formulations of Theorem 1, yet implies no argument whatsoever with [1]. Besides, [1] goes beyond previous publications and contains another important result, namely, the proof that if a system cannot be made SPR via constant feedback, no dynamic feedback can render it SPR. This note, therefore, only intends to present a brief history of early works and main developments that led to the complete proofs of this important result and do not seem to be too well known in the control community. Reference [1] served as a trigger for our note only because it was the last in a series of publications dealing with this issue.

II. BRIEF HISTORY

First, recall that a linear time-invariant system with a state-space realization $\{A_K, B, C\}$ is called SPR if there exist two positive definite matrices P and Q, such that the following two relations are simultaneously satisfied:

$$PA_K + A_K^T P = -Q \tag{1}$$

$$PB = C^T. \tag{2}$$

The first relation shows that a SPR system is asymptotically stable. The second relation yields $CB = B^T C^T$ (i.e., positive–definite symmetric) and implies that the transfer function $T(s)$ has relative degree m or, in other words, $T(s)$ has n poles and $n - m$ zeros.

The SPR property plays a crucial role in guaranteeing stability in systems with uncertainty [2], and in adaptive control [3]. However, as most real-world systems are not inherently SPR, the result presented as theorems above has proved very useful.

Based on the SPR representation (1)–(2) one can formally define the matrix $A = A_K + BKC$ with $K = 0$. It is known that the system $\{A, B, C\}$ remains SPR if one replaces $K = 0$ with any positive–definite and arbitrarily large gain. If one uses a negative–definite gain instead, one gets a system that is not necessarily SPR. However, such a system only needs a constant output feedback gain to become SPR. As output feedback affects the poles yet has no effect on the zeros of the system or on its input and output matrices, one is left with a new system $\{A, B, C\}$ that could be unstable, yet is minimum-phase and with $CB > 0$, like the SPR system [4]. Such systems that are separated only by constant output feedback from strict positive realness have been called "almost strictly positive real (ASPR)" [4] and many examples that satisfied the assumptions of the theorem above were shown to be ASPR and were successfully tested using adaptive control [4], [7]. Although some proofs of the ASPR property of Theorem 2 had been available in the Russian literature both for single-input–single-output (SISO) [5] and multivariable systems [6], they were not known and probably their importance not very well appreciated in the Western literature at the time. Here, some partial proofs based on simple root-locus and angle arguments have appeared in the context of SISO systems [7], [8] and the first proofs of the theorem for the general multivariable case were developed by Owens *et al.* [9] using multivariable root-loci and by Teixeira [10] using the state-space representation. (Teixeira also presented his proof in [11], which was submitted for publication in 1988. Unfortunately, although based on the available knowledge at the time should have been the first known state-space proof of the theorem, this paper was not published due to an oversight in the review process. A 1999 attempt by Fradkov to present a review on the topic ended with a similar result.) Various proofs of the theorem based on state-space representation have been subsequently published by Gu [12] and, as mentioned, by Huang *et al.* [1]. As already mentioned, Huang *et al.* [1]

Manuscript received December 16, 2002; revised January 23, 2003, September 8, 2003, January 26, 2004, and July 16, 2004. Recommended by Associate Editor E. Bai.

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Digital Object Identifier 10.1109/TAC.2004.837565

went even further in emphasizing the importance of the ASPR property, as Theorem 2 of their paper shows that if constant feedback can not make a system SPR, then there exists no proper dynamic feedback to render it SPR.

The reader may have a hard time following the proofs of the theorem, and even its general formulation and implications, as they were imbedded among many other results in the papers above, each one within the scope of its particular interests. Therefore, for the ease of reference, this note ends with a short and direct proof of Theorem 2 (which does make use of ideas used by others, yet ends with a direct and very streamlined version of the proof).

III. Proof of Theorem 2

The basic ideas of the following proof actually make use of such early works as [13] and [14] that had clarified the previously obscure relationship between the state-space representation of a system and the poles and zeros of its corresponding transfer function. First recall [13] that the zeros of the closed-loop system $\{A_K = A - BKC, B, C\}$ are identical with the zeros of the open-loop system $\{A, B, C\}$. Following [13], select the matrices $M_{n,n-m}, N_{n-m,n}$, such that $CM = 0, NB = 0$, and $NM = I$. Such matrices always exist [13] and the zeros of the system are then the eigenvalues of the matrix NAM. Because the system is strictly minimum phase, NAM is Hurwitz and there exists some positive–definite matrix P_0 such that

$$P_0(NAM) + (NAM)^T P_0 = -Q_0 < 0. \tag{3}$$

Before continuing, note that by using the relation $NM = I$ (not used in [1]), it is easy to see that the condition formulated in Theorem 1 is equivalent with (3) and makes Theorem 1 equivalent with Theorem 2. Now, it must be shown that there exists a matrix K that fulfils the conditions of Theorem 2. To this end, consider the matrix

$$P = N^T P_0 N + C^T (CB)^{-1} C. \tag{4}$$

Although P in (4) seems to be only positive–semidefinite symmetric, it is easy to show that it is actually positive–definite symmetric. Because $NB = 0$ implies that the rows of N and the columns of B are linearly independent, one can select the nonsingular matrix

$$N_B = \begin{bmatrix} N \\ B^T \end{bmatrix} \tag{5}$$

so that $P > 0$ if and only if $\tilde{P} = N_B P N_B^T > 0$. Here

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} = \begin{bmatrix} N \\ B^T \end{bmatrix} P [N^T \quad B] \tag{6}$$

$$\tilde{P}_{11} = NPN^T = NN^T P_0 NN^T + NC^T (CB)^{-1} CN^T \tag{7}$$

$$\tilde{P}_{12} = NPB = NC^T \tag{8}$$

$$\tilde{P}_{21} = B^T PN^T = CN^T \tag{9}$$

$$\tilde{P}_{22} = B^T N^T P_0 NB + B^T C^T (CB)^{-1} CB = B^T C^T. \tag{10}$$

We also compute

$$\tilde{P}_{12} \tilde{P}_{22}^{-1} \tilde{P}_{21} = NC^T (B^T C^T)^{-1} CN^T, \tag{11}$$

Therefore

$$\tilde{P}_{11} - \tilde{P}_{21} \tilde{P}_{22}^{-1} \tilde{P}_{12} = NN^T P_0 NN^T. \tag{12}$$

Thus, it is clear that $\tilde{P}_{11} > 0, \tilde{P}_{22} > 0$, and $\tilde{P}_{11} - \tilde{P}_{21}\tilde{P}_{22}^{-1}\tilde{P}_{12} > 0$, which imply that $\tilde{P} = N_B P N_B^T > 0$ and therefore $P > 0$.

It is easy to see now that the SPR relation (2) is satisfied. However, one must also show that (2) is satisfied for some positive definite gains K "sufficiently large." To clarify the meaning of "large" in this context, it is worth mentioning that in matrical case the notations $K_1 > K_2$ and $K_1 - K_2 > 0$ imply that $K_1 - K_2$ is positive definite.

Because $NB = 0$ one gets

$$\begin{aligned} PA_K + A_K^T P = {} & N^T P_0 NA + AN^T P_0 N \\ & + C^T (CB)^{-1} CA + A^T C^T (CB)^{-1} C \\ & - C^T KC - C^T K^T C. \end{aligned} \tag{13}$$

Again, it is not directly seen that (13) is negative definite. Therefore, consider the matrix

$$T = [M \quad B(CB)^{-1}]. \tag{14}$$

It is easy to see that T is square and invertible and its inverse is

$$T^{-1} = \begin{bmatrix} N \\ C \end{bmatrix}. \tag{15}$$

To show that (13) is negative definite, compute the expression

$$T^T \left(PA_K + A_K^T P \right) T = -Q. \tag{16}$$

Here

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{17}$$

$$Q_{11} = -[P_0(NAM) + (NAM)^T P_0] = Q_0 > 0 \tag{18}$$

$$Q_{12} = -P_0 NAB(CB)^{-1} - (CAM)^T (CB)^{-1} \tag{19}$$

$$Q_{21} = -(CB)^{-1} CAM - (NAB(CB)^{-1})^T P_0 = Q_{12}^T \tag{20}$$

$$\begin{aligned} Q_{22} = {} & K + K^T - (CB)^{-1} CAB(CB)^{-1} \\ & - (CAB(CB)^{-1})^T (CB)^{-1}. \end{aligned} \tag{21}$$

For K sufficiently large (positive definite), one gets

$$Q_{22} > 0 \tag{22}$$

and

$$Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} > 0. \tag{23}$$

Relations (18), (22), and (23) show that $Q > 0$ and, therefore, the closed-loop system is SPR. Note that if $K = kKo$, where k is scalar, there exists a limiting value ko for the gain k that would make Q positive semidefinite, thus defining a so-called "weakly" SPR system. This value can be considered a lower bound for the sought after gains, as any value $k > ko$ would make the system (strongly) SPR.

IV. Conclusion

This paper presented a brief history of the main contributions to an important SPR properties that led to the simple formulation of Theorem 2. A streamlined and direct proof of the theorem has also been presented. Present research seems to show that the procedure used in this note could eventually be extended to eliminate the symmetry condition from the SPR relations and also to extend the relations to nonstationary and nonlinear systems.

ACKNOWLEDGMENT

The author would like to thank A. Fradkov for starting this and helping to complete the brief history presented in this note, M. C. M. Teixeira for providing the first proof known by us and for bringing the work of Owens *et al.* [9] to our attention as a prior proof of the ASPR property in MIMO systems, and V. Balakrishnan for the useful hint that further streamlined the proof of this note.

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