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NUMERICAL ALGORITHMS FOR SQUARING–UP NON-SQUARE SYSTEMS PART II: GENERAL CASE

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Abstract

In this paper, we present numerical algorithms for squaring a non-square system by finding additional columns in input matrix (for tall systems) and by finding additional rows in output matrix (for fat systems). Several case are considered depending up the requirements on the rank of the input-output interaction matrix.

1. Introduction

In LQG/LTR controller design [1] of non-square systems, one is often forced to squaring the systems. This can be accomplished by squaring-up the system i.e., by finding additional *pseudo-inputs* or *pseudo-outputs* such that the resulting square system has its zeros at desired locations in the complex half plane. Squaring can also be accomplished by squaring the system down such that the resulting square system has minimum-phase [2]. However, it is well known that this is equivalent to solving an output feedback compensation problem and may typically require dynamic compensation, thereby increasing the order and complexity of the system.

This paper addresses the following problem: "Given the state matrix ($A \in R^{n\times n}$, system dynamics), the input matrix ($B \in R^{n\times m}$, location of actuators), the output matrix ($C \in R^{p\times n}$, location of the sensors) and the input-output interaction matrix $D \in R^{p\times m}$, $p \neq m$. Determine a pseudo-output matrix $\hat{C} \in R^{(m-p)\times n}$ and possibly an input-output interaction matrix $\hat{D} \in R^{(m-p)\times m}$ if $p < m$, such that the resulting square system has its zeros at the desired locations in the left half plane."

Note that, the problem of determining a pseudo-input matrix $\hat{B}$, when $m < p$ is the dual of the above problem and can be easily solved. In next two sections, we determine the conditions under which the above problem can be solved and develop a computational scheme for its solution.

Depending upon the dynamics of the given system, the following possible cases for selection of $\hat{C}$ and $\hat{D}$ may arise:

1) Input-output interaction matrix $D = O$.
   a) Augmented D should remain zero
   b) Augmented D should have rank $m - p$.
2) Rank of input-output interaction matrix $D = p$.
   a) Augmented D should have full rank
   b) Augmented D should have rank p.
3) Rank of input-output interaction matrix $D = r(< p)$.
   a) Augmented D should have rank $= r+m-p$.
   b) Augmented D should have rank r.

Of course to this list of possibilities we can add others where, for example, while the original D had full row rank, the augmented D matrix has rank less than m. However, such augmentation does not have any practical application. In the next section we outline the approach taken to determine the corresponding $\hat{C}$ and $\hat{D}$ to solve the various cases outlined above.

2. Input-Output Interaction Matrix $D = O$

In the sequel, we will discuss only the case when the system is *fat*, i.e., number of inputs is more than number of outputs. The case of *tall* systems is true by duality.

2.1. Augmented D remains zero

This case was treated in [4], however, for the sake of completeness the result is briefly summarized here. The following assumptions will be made on the system:

1) (A, B) is a controllable pair and B has full column rank ($= m$),
2) rank$(CB) = p$ (same as the number of outputs of the system).

For certain cases, additional requirements are put upon the system and they will be outlined at appropriate places.

Provided that Assumptions 1 and 2 above are satisfied, theoretically it is always possible to transform the system (A, B, C, O), by means of orthogonal state coordinate transformations, to the following form [4]:

$$\hat{S}(\lambda) = \left[\begin{array}{cc|c} A_{11}-\lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22}-\lambda I_{n-m} & O \\ \hline C_{11} & C_{12} & O \\ \hat{C}_{21} & \hat{C}_{22} & O \end{array}\right]. \quad (2.1)$$

where $[\hat{C}_{21}\ \hat{C}_{22}]$ are to be determined. By Assumption 2), the rank of C_{11} is p. Therefore $[\hat{C}_{21}\quad \hat{C}_{22}]$ can always be found such that rank of $\begin{bmatrix} C_{11} \\ \hat{C}_{21} \end{bmatrix}$ is m. Further, the system matrix can be written as

$$S(\lambda) \triangleq \left[\begin{array}{cc|c} A_{11}-\lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22}-\lambda I_{n-m} & O \\ \hline C_1 & C_2 & O \end{array}\right] \quad (2.2)$$

where, $[C_1\quad C_2] \triangleq \begin{bmatrix} C_{11} & C_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix}$. Now, let $\rho(\cdot)$ denote the rank of $(\cdot)$, then

$$\rho(S(\lambda)) = \rho \left[\begin{array}{cc|c} A_{11}-\lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22}-\lambda I_{n-m} & O \\ \hline C_1 & C_2 & O \end{array}\right]$$

$$= \mathrm{rank} \left[\begin{array}{cc|c} A_{11}-\lambda I_m & \times & B_1 \\ A_{21} & \tilde{A}_{22}-\lambda I_{n-m} & O \\ \hline C_1 & O & O \end{array}\right]. \quad (2.3)$$

where $\tilde{A}_{22} = A_{22} - A_{21}C_1^{-1}C_2$. Since, by construction, the rank of C_1 is m, clearly,

$$\rho(S(\lambda)) = 2m + \rho[\lambda I_{n-m} - A_{22} + A_{21}C_1^{-1}C_2]. \quad (2.4)$$

Further, $\mathrm{rank}(S(\lambda)) < n + m$ at all eigenvalues of the matrix $[A_{22} - A_{21}C_1^{-1}C_2]$. Knowing A_{22} and $A_{21}C_1^{-1}$, C_2 can be selected such that the matrix $A_{22} - A_{21}C_1^{-1}C_2$ has all its eigenvalues at desired locations in the left half plane. Equivalently, the problem of finding the *augmented* output matrix $[C_1\quad C_2]$ such that the system (A, B, C, O) is minimum phase, which in turn solves the problem of squaring up a non-square system, can be reduced to solving a state feedback problem [5], [6].

Example 2.1: For the sake of illustration, we consider a 6-th order system with 3 inputs and 2 outputs. The state space description of the system is characterised by

$$A = \begin{bmatrix} 3 & 4 & 1 & 3 & 0 & 3 \\ 3 & 5 & 2 & 1 & 1 & 1 \\ 3 & 4 & 2 & 0 & 2 & 2 \\ 3 & 2 & 0 & 4 & 4 & 5 \\ 2 & 1 & 3 & 0 & 5 & 4 \\ 4 & 5 & 4 & 4 & 1 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 3 & 1 \\ 2 & 0 & 5 \\ 3 & 5 & 3 \\ 4 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 1 & 2 & 5 & 1 & 3 \\ 4 & 4 & 1 & 4 & 4 & 5 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For this system, using the results outlined about, to assign the transmission zeros of the squared system at -1, -2 and -3, the required third vector in the output matrix was found to be

$c = [0.3542\ \ -0.4597\ \ 0.0342$
$1.1739\ \ 0.2095\ \ 0.7600]$

and the input-output matrix was a null matrix of order 3. The transmission zeros were found to be at the desired locations.

2.2. Rank of Augmented D should be $m - p$

While in the previous case the squared system could have only $n - m$ transmission zeros, the number of zeros assignable while squaring the system increases to $n - p$. The augmented system can be transformed by means of orthogonal state coordinate transformations to the following form:

$$U(\lambda)S(\lambda)V(\lambda)$$

$$= \left[\begin{array}{cc|cc} A_{11}-\lambda I_{n-p} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22}-\lambda I_p & O & B_{22} \\ \hline C_{11} & C_{12} & O & O \\ \hat{C}_{21} & \hat{C}_{22} & O & \hat{D}_{22} \end{array}\right] \quad (2.5)$$

where $U(\lambda)$ and $V(\lambda)$ are unimodular matrices and the matrices $\hat{C}_{21}$, $\hat{C}_{22}$ and $\hat{D}_{22}$ need to be determined. In order for the transmission zeros to be assigned, we need the following assumption on the transformed system: The rank both of C_{11} as well as B_{11} should be p. Now, since rank of C_{11} and B_{11} is p, applying some block row and column operations, it can be shown that

$$\rho(S(\lambda)) = 2p + \rho \left[\begin{array}{c|c} A_{22} - A_{21}C_{11}^{-1}C_{12} - \lambda I_{n-p} & B_{22} \\ \hline \hat{C}_{22} & \hat{D}_{22} \end{array}\right]$$

$$\triangleq \rho \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C}_{21} & \hat{D}_{22} \end{array}\right] \quad (2.6)$$

Since we can always select $\hat{D}_{22}$ to be a full rank matrix, if $(\hat{A}, \hat{B})$ is a controllable pair, then all $n - p$

transmission zeros of the augmented system can be assigned at desired locations.

Example 2.2: Consider the same system as in Example 2.1. In this case $\hat{D}$ is required to have rank $(m-p)$. For sake of simplicity, the augmented system was chosen to have

$$\hat{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With this choice of $\hat{D}$, to assign the squared system's transmission zeros at -1, -2, -3 and -4, the additional output vector was found to be

$$c = [-3.6037 \;\; -24.073 \;\; -4.3615 \;\; 17.759 \;\; -2.1623 \;\; 7.9220].$$

It should be noted that, (to the best of the author's knowledge) the assumptions on the transformed system cannot be stated in terms of any specific assumptions on the original system.

3. Rank of Input-Output Matrix D is p

Here again, we will consider two cases. First, where the rank of augmented input-output matrix is changed to the maximum possible and second where the D matrix of the squared system is left unchanged.

3.1. Augmented D should have rank m

This case is fairly straightforward. Using singular value decomposition, it is always possible to find $\hat{D}$ such that the augmented D has full rank $(= m)$. Once that is done, all n transmission zeros of the squared system can be assigned by solving the state feedback problem $A - BD^{-1}C$, where C is partly unknown. This is a trivial case and therefore, we have not included an example to illustrate it.

3.2. Augmented D should have rank p

In this case, it is required that $\hat{D} = O$. Again assume that using unimodular transformations $U(\lambda)$ and $V(\lambda)$, the system has been transformed to $U(\lambda)S(\lambda)V(\lambda) =$

$$\left[\begin{array}{cc|cc} A_{11}-\lambda I_{n-p} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22}-\lambda I_p & O & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & O \\ \hat{C}_{21} & \hat{C}_{22} & O & O \end{array}\right]. \qquad (3.1)$$

Using D_{11} as a pivot, we get

$$\rho(S(\lambda)) = p + \rho \left[\begin{array}{cc|c} \tilde{A}_{11}-\lambda I_{n-p} & \tilde{A}_{12} & B_{12} \\ A_{21} & A_{22}-\lambda I_p & B_{22} \\ \hline \hat{C}_{21} & \hat{C}_{22} & O \end{array}\right] \qquad (3.2)$$

$$\triangleq \rho \left[\begin{array}{c|c} \hat{A}-\lambda I_n & \hat{B} \\ \hline \hat{C} & O \end{array}\right].$$

Now, provided $(\hat{A}, \hat{B})$ is a controllable pair, and $\hat{B}$ has full column rank, this case can be resolved in a manner similar to that treated in Section 2.1., i.e., we first perform row compression on $\hat{B}$ and then using block row and column operations determine the square invertible subsystem whose zeros will be the transmission zeros of the original system. Further, as before, these zeros can be assigned at desired locations in the complex plane.

Example 3.1: Once again, the same system triple (A, B, C) as in Example 2.1 is considered. However, the D matrix (again for sake of simplicity of presentation) is selected to be

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Is clear that the compensated system has five transmission zeros. The locations of the desired transmission zeros were selected to be -1, -2, -3, -4 and -5. Using the results presented above, it was found that if C is augmented by

$$c = [4.5304 \;\; 12.840 \;\; 5.3000 \;\; -7.5324 \;\; -3.9470 \;\; -7.7026],$$

then the resulting squared system will have the desired set of transmission zeros. Note that in this case, the augmented input output matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. Rank of $D = r (< p)$

This case is a combination of results developed in Sections 2. and 3. Here again, we will consider the following two situations: First, the rank of input-output matrix of the squared system is maximum (in this case $\text{rank}(D) = m + r - p$. The resulting D matrix remains rank deficient. Second, where D is left unchanged.

4.1. Augmented D has rank $m + r - p$

Since D is assumed to have rank r $(< p)$, the given system can be transformed by means of state coordinate transformations to the following form: $U(\lambda)S(\lambda)V(\lambda) =$

$$\left[\begin{array}{cc|ccc} A_{11}-\lambda I_{n-r} & A_{12} & B_{11} & B_{12} & B_{12} \\ A_{21} & A_{22}-\lambda I_r & O & B_{22} & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & O & O \\ C_{21} & C_{22} & O & O & O \\ \hat{C}_{31} & \hat{C}_{32} & O & O & \hat{D}_{22} \end{array}\right]. \qquad (4.1)$$

Performing some block row and block column operations on (4.1), it can be shown that $\rho(\mathcal{S}(\lambda)) =$

$$r+\rho\left[\begin{array}{cc|cc} \tilde{A}_{11}-\lambda I_{n-r} & \tilde{A}_{12} & B_{12} & B_{13} \\ A_{21} & A_{22}-\lambda I_r & B_{22} & B_{23} \\ \hline C_{21} & C_{22} & O & O \\ \hat{C}_{31} & \hat{C}_{32} & O & \hat{D}_{33} \end{array}\right].$$

where, $\tilde{A}_{11} = A_{11} - B_{11}D_{11}^{-1}C_{11}$ and $\tilde{A}_{12} = A_{12} - B_{11}D_{11}^{-1}C_{12}$. At this stage the situation becomes identical to that studied in Section 2.2. Hence using the results from there, we can assign $n-m+r$ transmission zeros at desired locations. Specifically, let the system matrix be defined as

$$\tilde{\mathcal{S}}(\lambda)\left[\begin{array}{cc|cc} \tilde{A}_{11}-\lambda I_{n-r} & \tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22}-\lambda I_r & \tilde{B}_{22} & \tilde{B}_{23} \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & O & O \\ \hat{C}_{31} & \hat{C}_{32} & O & \hat{D}_{33} \end{array}\right].$$

Now, assuming that $\rho[\tilde{C}_{11},\ \tilde{C}_{12}][\tilde{B}_{11}^T,\ \tilde{B}_{21}^T]^T$ has rank $(p-r)$, we can apply the results of Section 2.1. to $\tilde{\mathcal{S}}(\lambda)$ and get the desired output submatrix that will square the system while assigning $(n-m+r)$ transmission zeros at desired locations.

Since the results are practically the same as those presented in Sections 2 and 3, we are not including numerical examples for this case.

4.2. Augmented D has rank r

Clearly, in this case the D matrix is left unchanged. Looking at the transformed system matrix $\mathcal{U}(\lambda)\mathcal{S}(\lambda)\mathcal{V}(\lambda) =$

$$\left[\begin{array}{cc|ccc} A_{11}-\lambda I_{n-r} & A_{12} & B_{11} & B_{12} & B_{12} \\ A_{21} & A_{22}-\lambda I_r & O & B_{22} & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & O & O \\ C_{21} & C_{22} & O & O & O \\ \hat{C}_{31} & \hat{C}_{32} & O & O & O \end{array}\right]. \tag{4.2}$$

it is easy to see that $\rho(\mathcal{S}(\lambda)) = r+$

$$\rho\left[\begin{array}{cc|cc} \tilde{A}_{11}-\lambda I_{n-r} & \tilde{A}_{12} & B_{12} & B_{13} \\ A_{21} & A_{22}-\lambda I_r & B_{22} & B_{23} \\ \hline C_{11} & C_{12} & O & O \\ \hat{C}_{21} & \hat{C}_{22} & O & O \end{array}\right]. \tag{4.3}$$

where $\tilde{A}_{11} = A_{11} - B_{11}D_{11}^{-1}C_{11}$ and $\tilde{A}_{12} = A_{12} - B_{11}D_{11}^{-1}C_{12}$. Therefore, if the reduced order subsystem (4.3) satisfies the conditions in Section 2.1., the $(n-m+r)$ transmission can be assigned at the desired locations in the complex plane, subject to complex conjugate pairing.

5. Computation of $\hat{C}$ and $\hat{D}$

Here we show the computation of the $\hat{C}$ for one of the cases. The remaining cases, apart from state transformations follow much the same procedure. Under the assumption that rank of $C_{11} = p$, it is easy to see that C_{21} can be chosen such that the matrix C_1 has full rank. Any C_{21} lying in the null space of C_{11} will accomplish this goal. Numerical algorithms such as singular value decomposition can be employed to determine C_{21}.

Determination of C_2 is not so straightforward. Note that it is required that the matrix $A_{22} - A_{21}C_1^{-1}C_2$ have all its eigenvalues at desired locations. To see how this may be accomplished, let us write $C_2 := \hat{C}_2 + \tilde{C}_2$, where

$$\tilde{C}_2 = \begin{bmatrix} C_{12} \\ O_{(m-p)\times(n-m)} \end{bmatrix} \quad \text{and} \quad \hat{C}_2 = \begin{bmatrix} O_{p\times(n-m)} \\ C_{22} \end{bmatrix}, \tag{5.1}$$

where the subscript of O denotes its dimension and $C_{22} \in R^{(m-p)\times(n-m)}$.

Next, let $\tilde{A}_{22} \triangleq A_{22} - A_{21}C_1^{-1}\tilde{C}_2$. Then, the problem of determining $\hat{C}_{22}$ reduces to finding a state feedback matrix C_{22} such that the matrix $\tilde{A}_{22} - A_{21}C_1^{-1}\hat{C}_2$ has desired eigenvalues, where $\hat{C}_2 = \begin{bmatrix} O_{p\times(n-m)} \\ \hat{C}_{22} \end{bmatrix}$.

The above problem can be solved provided the subsystem $(A_{22},\ A_{21})$ is controllable. Note that the original system is assumed to be controllable. It is well known that for a controllable system

$$\text{rank}\left[\begin{array}{cc|c} A_{11}-\lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22}-\lambda I_{n-m} & O \end{array}\right] = n. \tag{5.2}$$

Knowing that rank $B_1 = m$ by assumption, the rank of $[A_{21},\ A_{22}-\lambda I_{n-m}]$ must be $n-m$. Equivalently (A_{22}, A_{21}) is a controllable pair. Therefore, it is always possible to find a C_{22} and hence $\hat{C}_2$ such that

$$\mathcal{S}(\lambda) = \left[\begin{array}{cc|c} A_{11}-\lambda I & A_{12} & B_1 \\ A_{21} & A_{22}-\lambda I & O \\ \hline C_{11} & C_{12} & O \\ C_{21} & C_{22} & O \end{array}\right] \tag{5.3}$$

has its zeros at the desired location in the left half plane. It should be emphasised that if the given system did possess any transmission zeros, they can be reassigned to the original locations (if required) by state feedback represented by $\tilde{A}_{22} - A_{21}C_1^{-1}\hat{C}_2$.

6. Concluding Remarks

In this paper, we discussed the problem of squaring-up an non-square system. Since squaring-up can be transformed into certain state feedback problems, it has some practical benefits over squaring the system down where the problem can be equated to an output feedback problem (often requiring dynamic compensation for complete solution). Numerical examples were provided to illustrate the proposed results.

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