NORMS

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This is a short document on the norms we will be using in this class. A norm on a vector space V, usually denoted as $\| \ \|$, maps from a vector space V to the real numbers and is always greater than or equal to zero. Our vector spaces of interest are $\mathbb{R}^n$ (vectors containing all real elements) and possibly $\mathbb{C}^n$ (vectors of complex elements).

Definition 1 (Simple norm definition)**.** Let $a \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. All norms satisfy the following three properties.

(a) $\|x\| = 0$ if and only if $x = 0$
(b) $\|ax\| = |a| \, \|x\|$
(c) $\|x + y\| \leq \|x\| + \|y\|$

For those more mathematically inclined I give the broader definition.

Definition 2 (Formal norm definition)**.** V is a vector space over the field K. Let the norm be defined as $\|\cdot\| : V \longrightarrow [0, \infty)$. Let $a \in K$ and $x, y \in V$. All norms satisfy (a), (b) and (c) from the above definition.[1]

1. Vector Norms in the Reals

With the following subscript definition,

$$x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

for all $x \in \mathbb{R}^n$, $1 \leq p < \infty$

$$\|x\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

and

$$\|x\|_\infty \triangleq \sup_i |x_i| \, .$$

The norms are referred to as p-norms (1-norm, 2-norm, . . ., ∞-norm). It will always be assumed that when the subscript on the p-norm is not given, we are assuming $p = 2$,

$$\|x\| \triangleq \|x\|_2.$$

A visualization of these norms is given in Figure 1.

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[1] The interested reader can learn more about normed spaces on MIT OpenCourseWare 18.102 Functional Analysis.

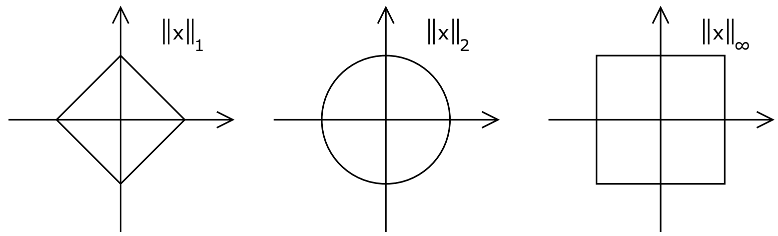

Figure 1. Adapted from http://en.wikipedia.org/wiki/Norm_(mathematics)

2. Matrix Norms

Let $A \in \mathbb{R}^{m\times n}$ then the induced p-norm is defined as

$$\|A\|_p \triangleq \sup_{\|x\|=1} \frac{\|Ax\|_p}{\|x\|_p}.$$

when the Euclidean norm is used, i.e. $p = 2$ we have that

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

where $\lambda_{\max}$ denotes the maximum Eigenvalue. Also as before, when no p value is denoted, it is assumed that $p = 2$,

$$\|A\| = \|A\|_2.$$

Property 1. *The induced p-norms have two special properties. Let* $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times d}$

$$\begin{aligned} \|Ax\|_p &\le \|A\|_p \|x\|_p \\ \|AB\|_p &\le \|A\|_p \|B\|_p. \end{aligned} \tag{1}$$

Note that this property need not hold for all matrix norms.

3. Vector Signal Norms

Let $x(t) : \mathbb{R} \to \mathbb{R}^n$ and $1 \le p < \infty$

$$\|x(t)\|_{L_p} \triangleq \left(\lim_{t\to\infty} \int_0^t \|x(\tau)\|^p d\tau \right)^{1/p}. \tag{2}$$

and

$$\|x(t)\|_{L_\infty} \triangleq \sup_t \|x(t)\| \tag{3}$$

Note that we are not using the essential supremum here, but just the supremum. If the essential supremum were being used, then we would allow the signal to be unbounded on a set of measure zero, and it would still be in $\mathcal{L}_\infty$. With our definition, if a signal is in $\mathcal{L}_\infty$ then the signal is bounded for all time.

The above norm is called the L_p-norm. The p-norm was used on a time varying signal $x(t)$ giving us a time dependent "norm" $\|x(t)\|$. At each fixed t, $\|x(t)\|$ is a norm, but it is not true to say that for all t, $\|x(t)\|$ is a norm. We need a scalar value for all t, so we resorted to integration or using the supremum operator.

It is notationally equivalent to write L_p, L^p, $\mathcal{L}_p$, $\mathcal{L}^p$. Its just a preference of superscript, subscript, and or calligraphic text.

Holder's Inequality. *Let f(t) and g(t) be scalar functions of time with bounded L_p and L_q norms respectively where $1 = 1/p + 1/q$, then*

$$\|f(t)g(t)\|_{L_1} \leq \|f(t)\|_{L_p}\|g(t)\|_{L_q} \tag{4}$$

Holder's inequality will be used in the derivation of the L_1-norm for $\dot{\theta}(t)$ in lectures 3 and 4.

Cauchy-Schwarz Inequality. *Holder's inequality with $p = 2$.*

Example 1. Consider the time response of the dynamical system $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$,

$$\dot{x}(t) = Ax(t) + bu(t) \tag{5}$$

where A is Hurwtiz and further more we are told that $u(t) \in L_2$, i.e. there exists a $c_1 \geq 0$ such that$\|u(t)\|_{L_2} \leq c_1 < \infty$. Given that A is hurwitz, there exists $c_2, c_3 > 0$ such that

$$\|\exp(At)\| \leq c_2 \exp(-c_3 t). \tag{6}$$

Details on bounding the matrix exponential can be found in [1]. The time series response of $x(t)$ is

$$x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-\tau))u(\tau)d\tau \tag{7}$$

Using the bound in (6) and taking the 2-norm for fixed t we have

$$\|x(t)\| \leq c_2\text{exp}(-c_3 t)\|x(0)\| + \int_0^t c_2 \exp(-c_3(t-\tau))\|u(\tau)\|d\tau. \tag{8}$$

Using Holders inequality on the last term we have that

$$\|x(t)\| \leq c_2\text{exp}(-c_3 t)\|x(0)\| + \sqrt{\int_0^t c_2^2 \exp(-2c_3(t-\tau))d\tau}\sqrt{\int_0^t \|u(\tau)\|^2 d\tau}. \tag{9}$$

Using the bound given to us for $u(t)$ we can say

$$\|x(t)\| \leq c_2\text{exp}(-c_3 t)\|x(0)\| + c_1 c_2\sqrt{\int_0^t \exp(-2c_3(t-\tau))d\tau}. \tag{10}$$

Gronwall Bellman Lemma. *For $u, v \geq 0$ and c_1 a positive constant, and if*

$$u(t) \leq c_1 + \int_0^t u(\tau)v(\tau)d\tau \tag{11}$$

then

$$u(t) \leq c_1 \exp\left(\int_0^t v(\tau)d\tau\right).$$

Proof. From (11) we have
$$\frac{u(t)v(t)}{c_1+\int_0^t u(\tau)v(\tau)d\tau} \leq v(t).$$
Integrating both sides between 0 and τ we have
$$\log\left(c_1+\int_0^t u(\tau)v(\tau)d\tau\right) - \log c_1 \leq \int_0^t v(\tau)d\tau.$$
Adding $\log c_1$ to both sides and taking the exponent we have
$$u(\tau) \leq c_1 + \int_0^t u(\tau)v(\tau)d\tau \leq c_1 \exp\left(\int_0^t v(\tau)d\tau\right). \qquad \square$$

Lemma 2 (scalar version of Lemma 2.2 in [2])**.** *The scalar dynamical system described by*
$$\dot{x} = (a+b(t))x \tag{12}$$
with $a<0$, $b \in \mathcal{L}_2$ results in bounded trajectories for x.

Proof. The solutions to (12) is
$$x(t) = \exp(at)x(0) + \int_0^t \exp(a(t-\tau))b(\tau)x(\tau)d\tau.$$
First note that $\exp(at)x(0) \leq x(0)$, then we can conclude that
$$x(t) \leq x(0) + \int_0^t \exp(a(t-\tau))b(\tau)x(\tau)d\tau,$$
and taking 2-norms
$$\|x(t)\| \leq \|x(0)\| + \int_0^t \|\exp(a(t-\tau))b(\tau)\|\|x(\tau)\|d\tau,$$
Applying Gronwall-Bellman Lemma where
$$\begin{aligned} c_1 =& \|x(0)\| \\ u =& \|x(\tau)\| \\ v =& \|\exp(a(t-\tau))b(\tau)\| \end{aligned}$$
results in
$$\begin{aligned} \|x(t)\| \leq& \|x(0)\| \exp\left(\int_0^t \|\exp(a(t-\tau))b(\tau)\|d\tau,\right) \\ \leq& \|x(0)\| \exp\left(\int_0^t \|\exp(a(t-\tau))\|\|b(\tau)\|d\tau,\right) \end{aligned}$$
Application of Cauchy Schwarz inequality results in
$$\|x(t)\| \leq \|x(0)\| \exp\left(\sqrt{\int_0^t \|\exp(a(t-\tau))\|^2 d\tau}\sqrt{\int_0^t \|b(\tau)\|^2 d\tau}\right).$$

The quantity $\int_0^t \|\exp(a(t-\tau))\|^2 d\tau \leq \frac{1}{2|a|}$ and we are told that $b \in \mathcal{L}_2$. Thus

$$\|x(t)\| \leq \|x(0)\| \exp\left(\sqrt{\frac{1}{2\,|a|}}\|b(t)\|_{L_2}\right). \qquad \square$$

References

1. C. Moler and C.V. Loan, *Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later*, Siam Review (2003).
2. K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems*, Dover, 2005.

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